The Emergence of Spacetime:
Entanglement & Complexity

Robert Myers
Lecture 5

"It from Qubit": New Collision of Ideas

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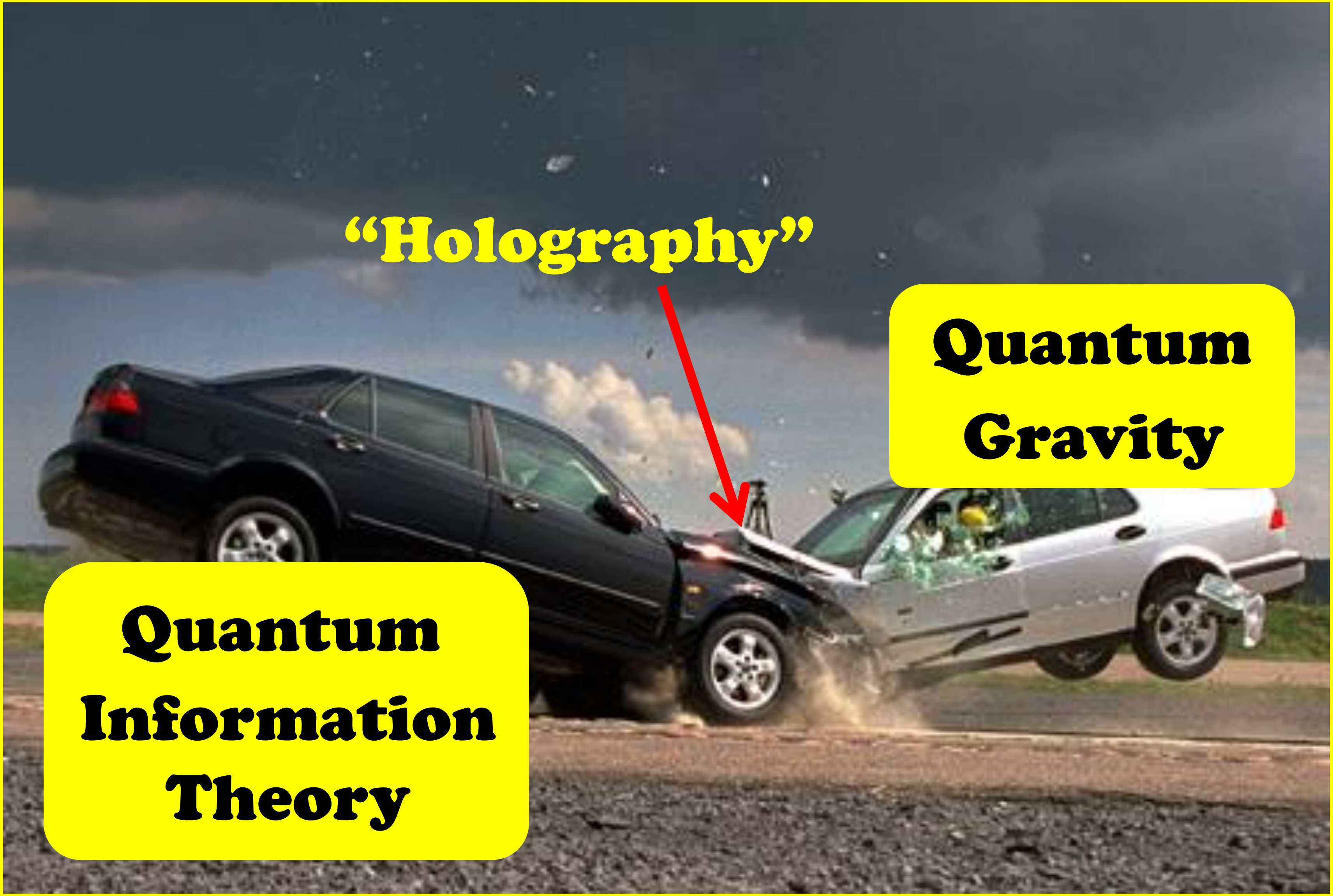
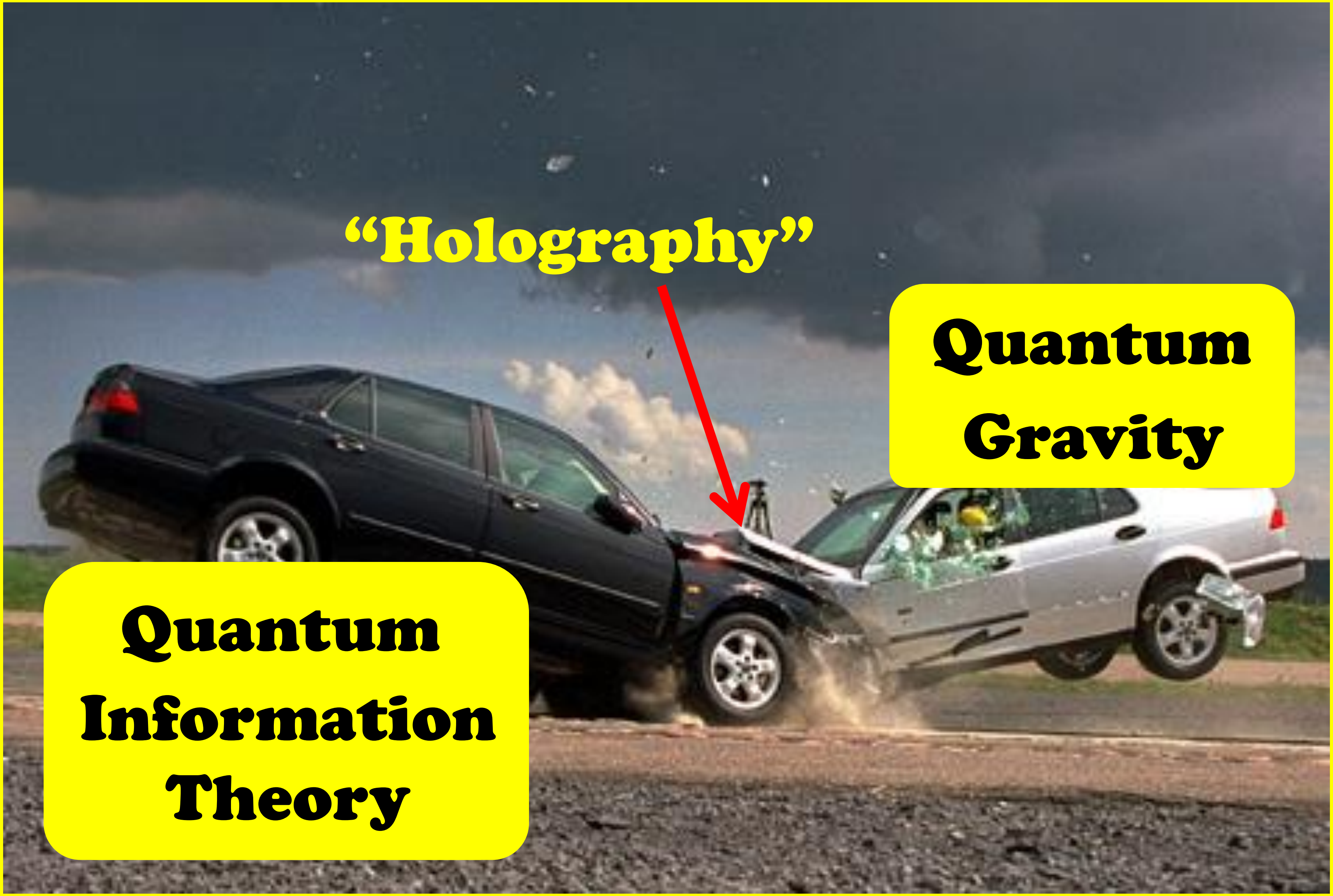


"It from Qubit": New ~~Collision~~ ~~Confluence~~ ~~Nexus~~ Dialogue of Ideas



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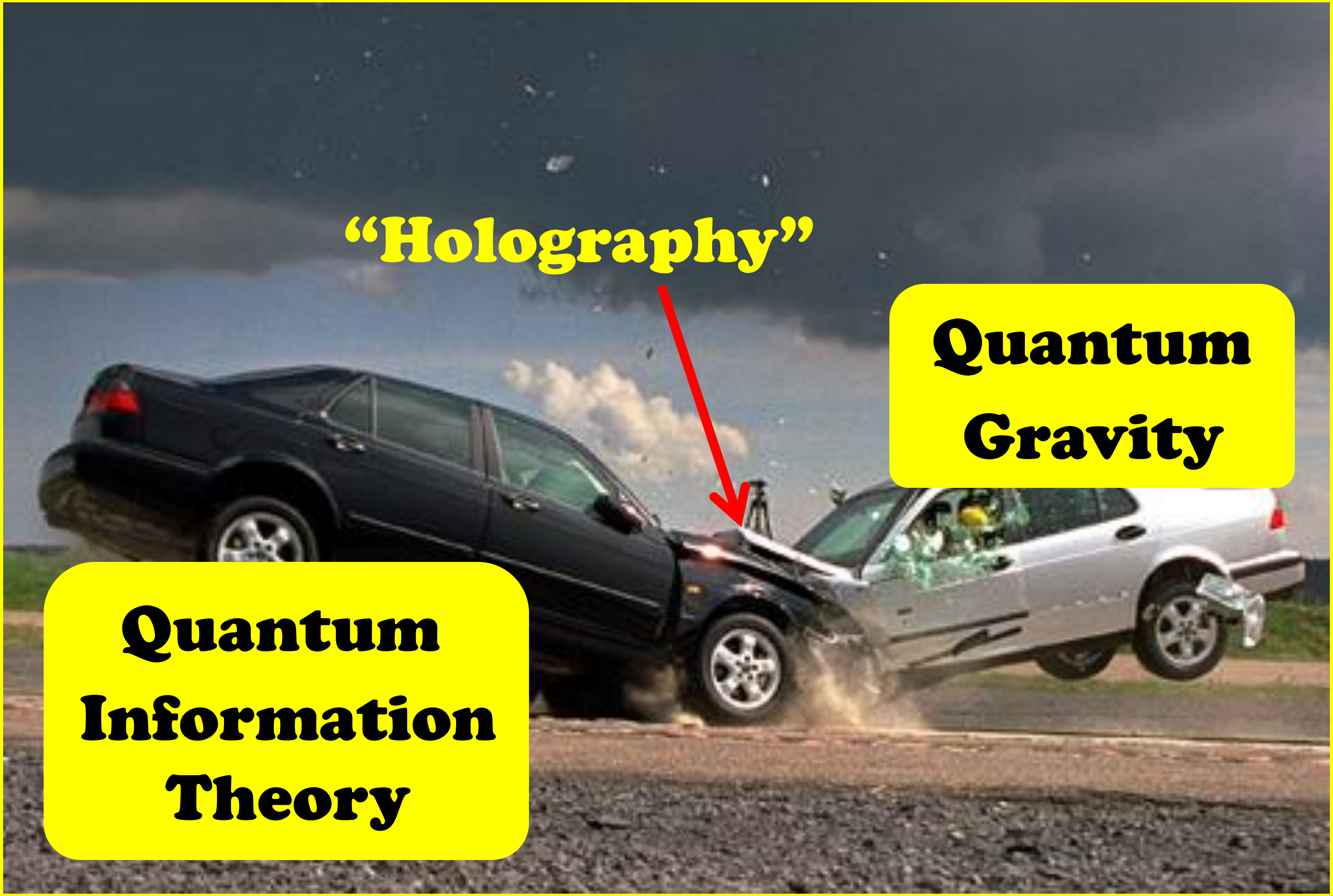
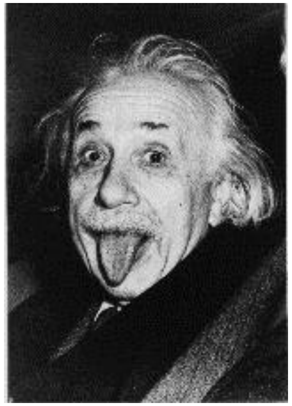
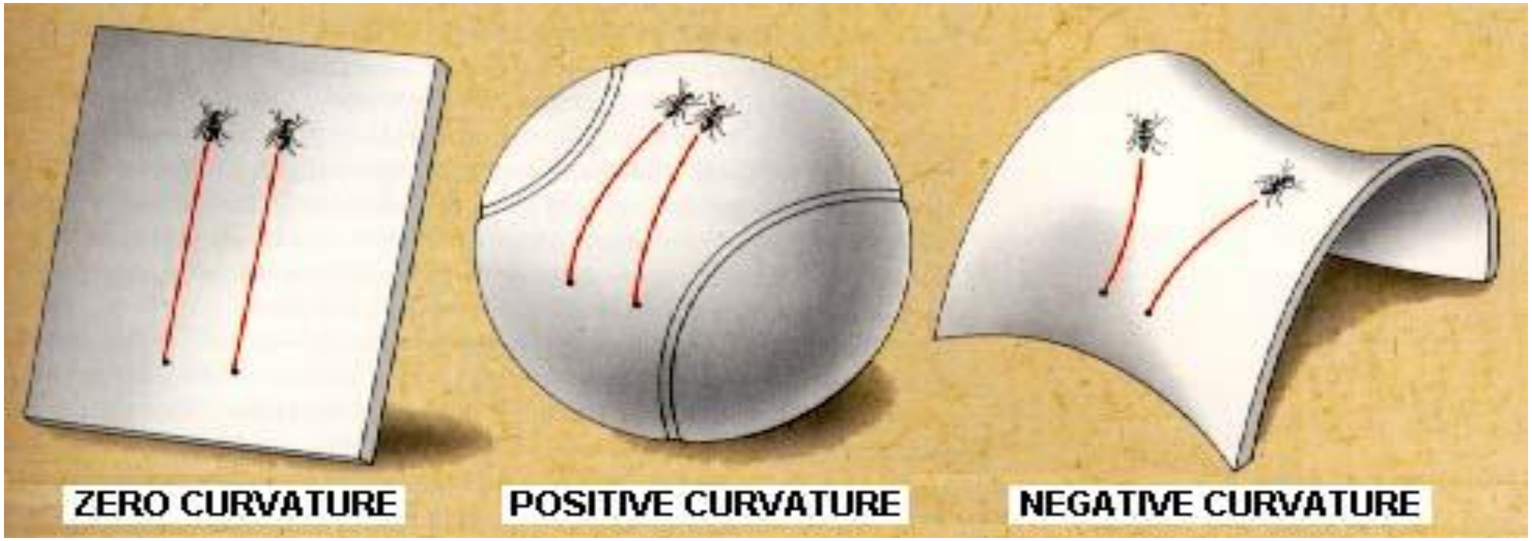


Einstein:

Gravity? It's all about geometry!





General Relativity is the geometric arena for physics on very large scales: planets, stars, galaxies, cosmology

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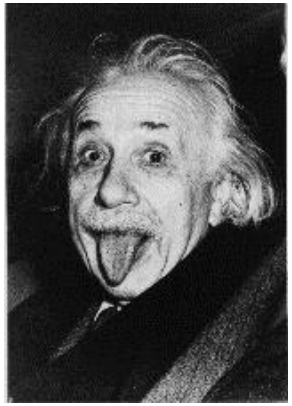
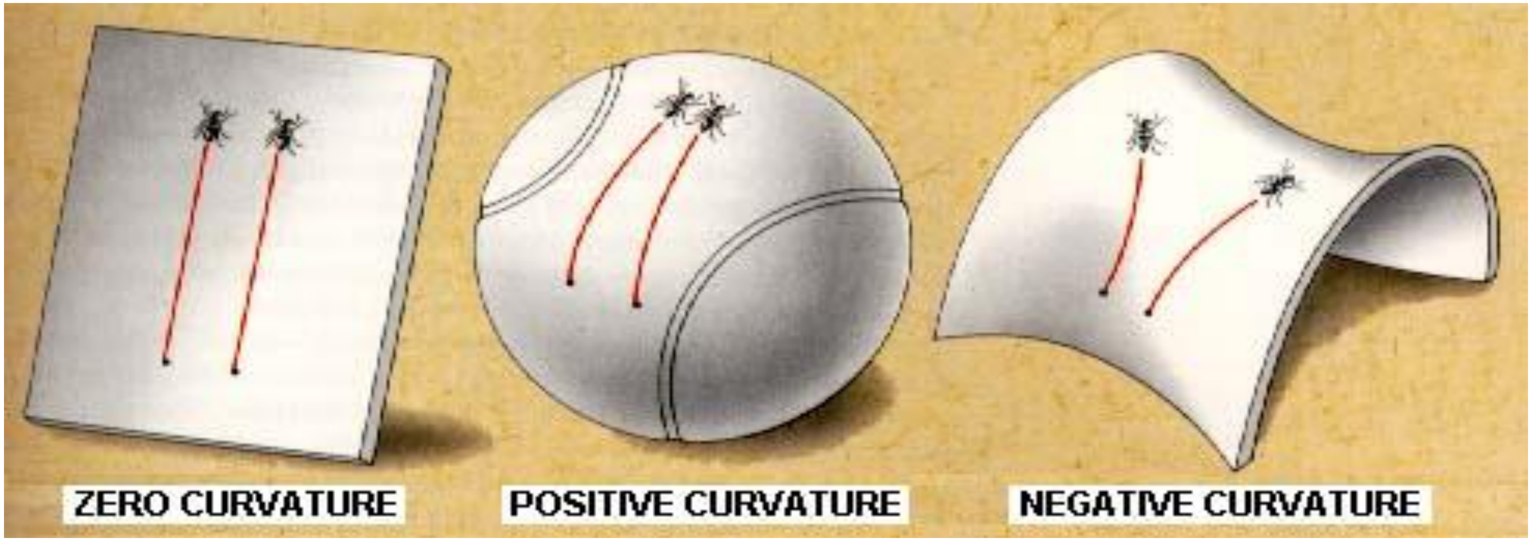
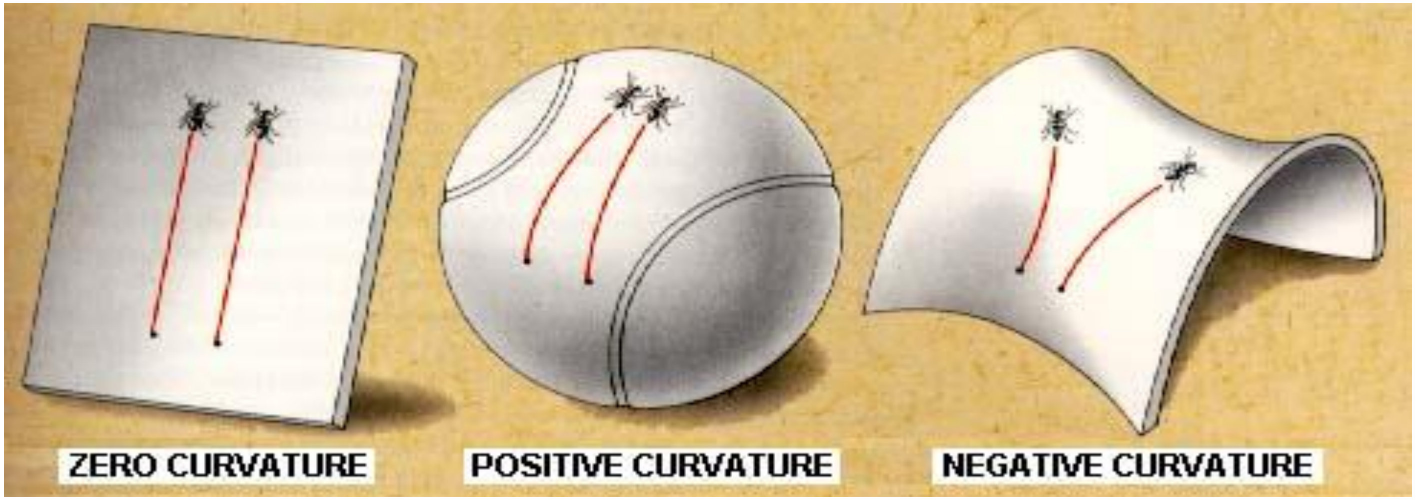




Spacetime moves from simply stage for physical phenomena, to being both the stage and an active player in the dynamics

General Relativity + Quantum Theory = Quantum Gravity?



$$+\,\hbar$$

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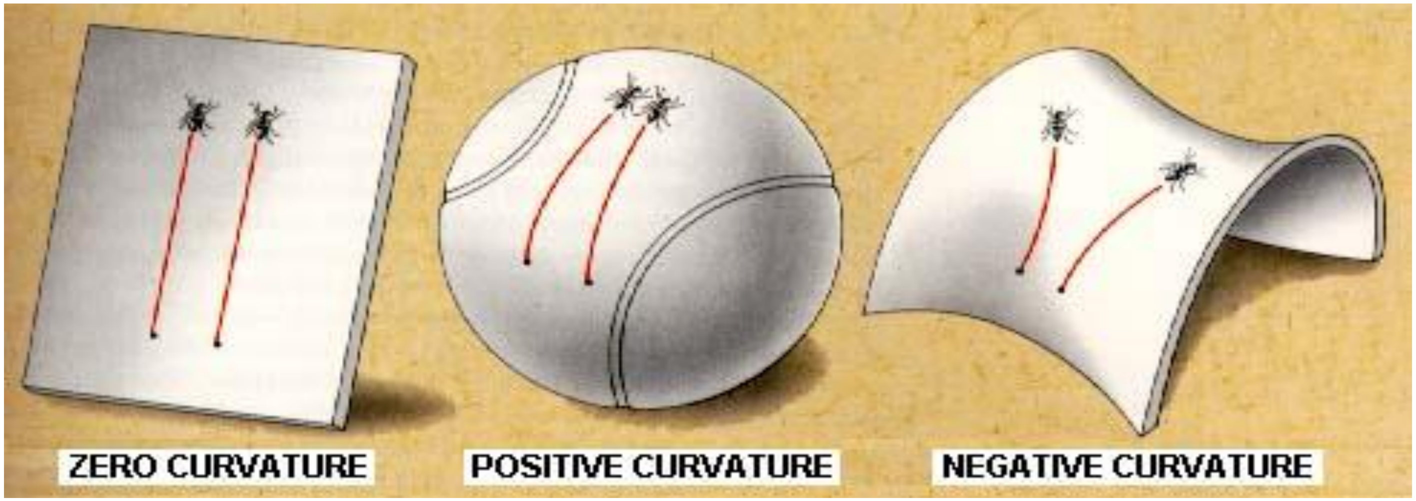
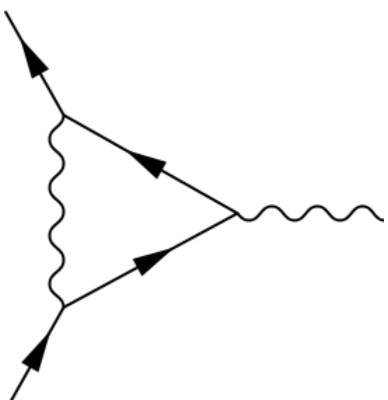


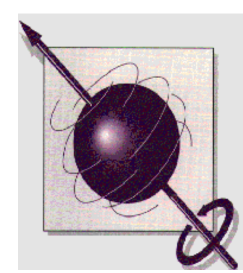
$+\,\hbar$

- quantum fluctuations become manifest at small scales e.g., magnetic moment of the electron, $\mu_e = g\, e\hbar/4m_e$, with $g \approx 2$ but modified by quantum fluctuations



$$g_{\text{theory}} = 2.002319304\boxed{3070}$$

$$\left|\frac{\mu_{\text{theory}} - \mu_{\text{experiment}}}{\mu_{\text{experiment}}}\right| \leq 10^{-10}$$



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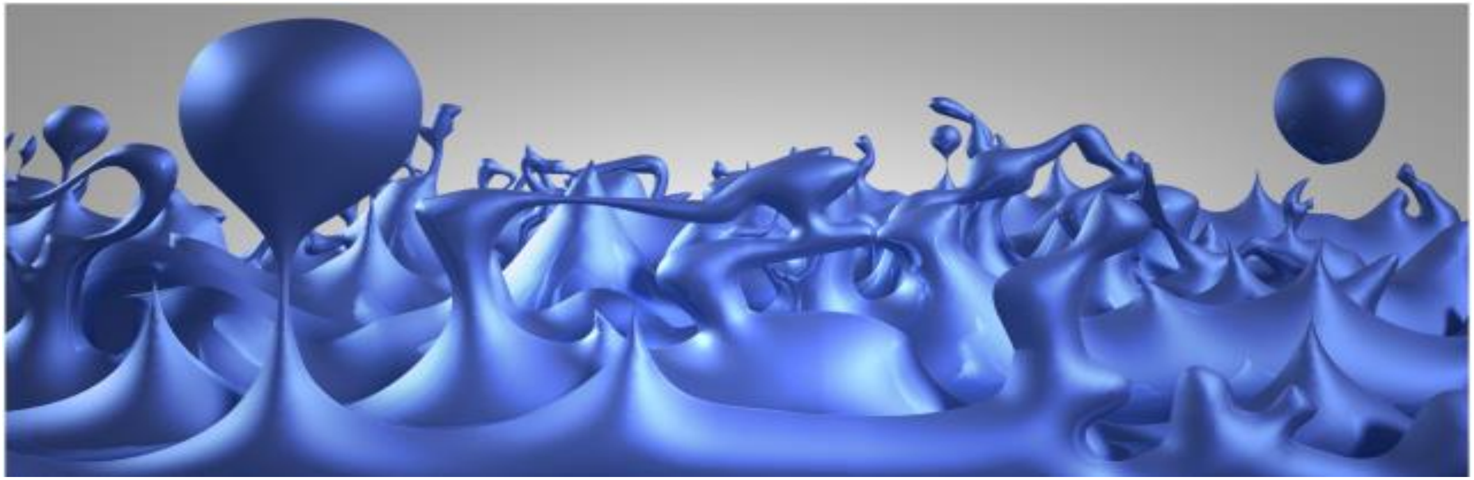

- spacetime geometry exhibits strong fluctuations when examined on very short distance scales
- how do we make sense of spacetime framework?

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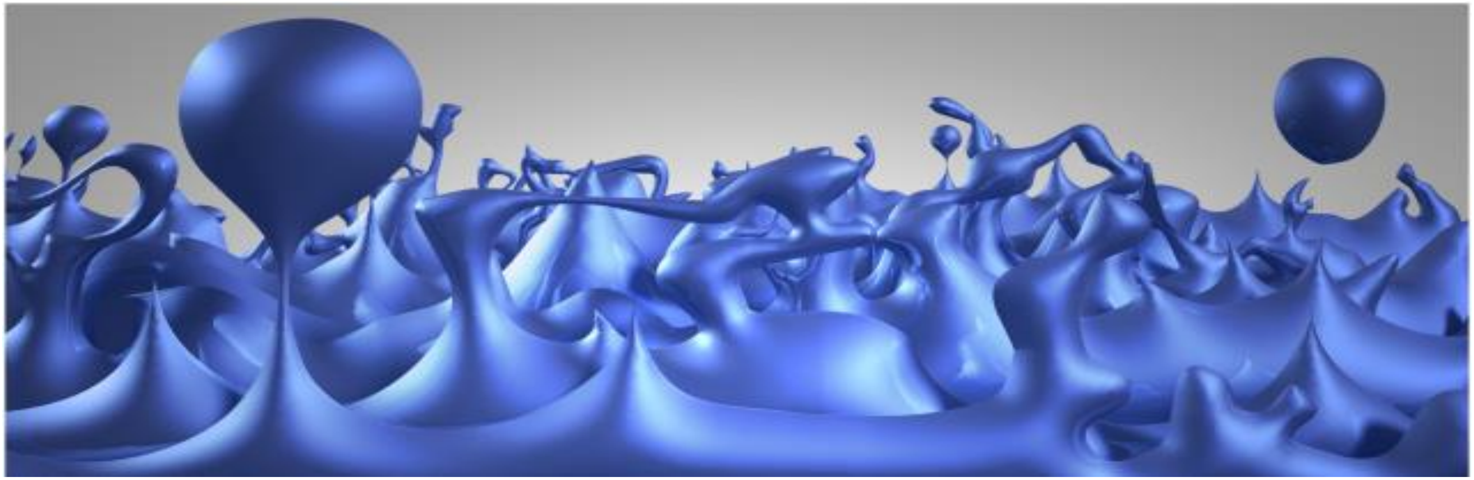

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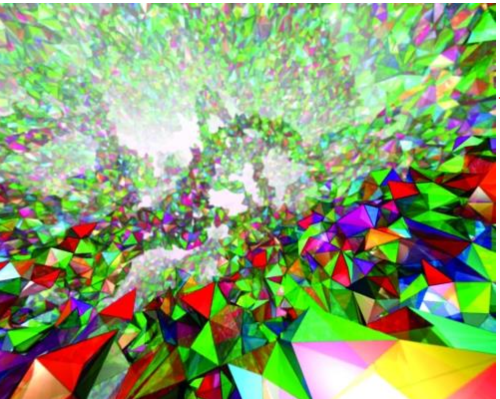
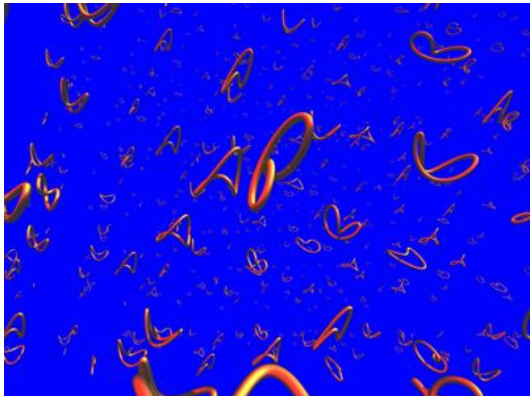


modify geometry
at short distances

modify spectrum
at short distances



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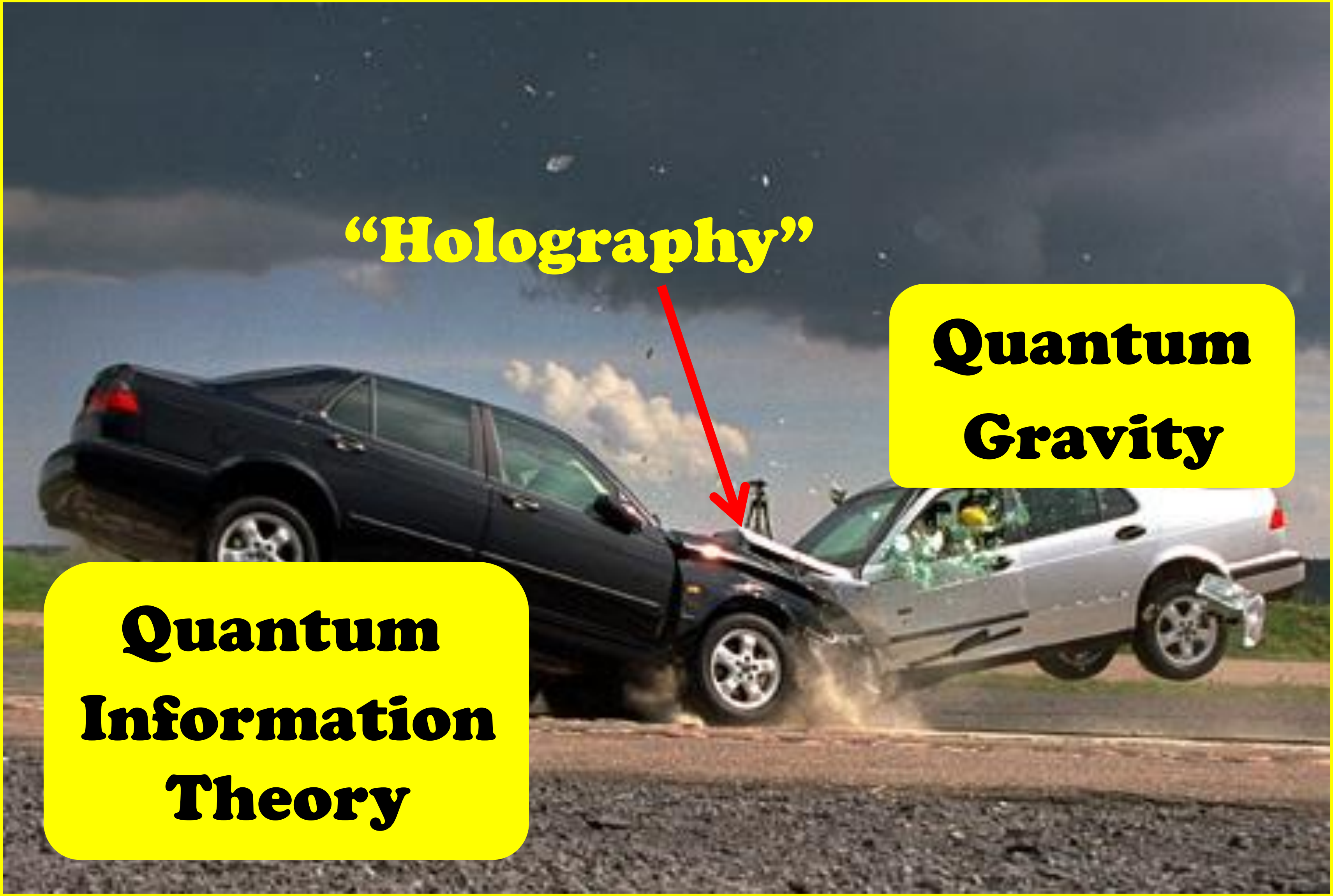
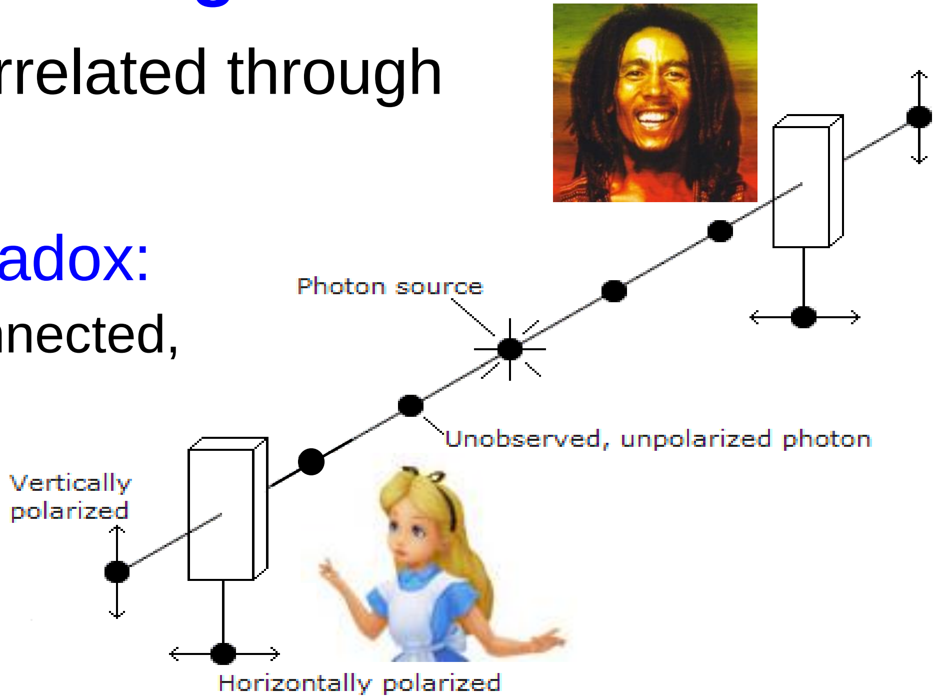


Quantum Entanglement

- different subsystems are correlated through global state of full system

Einstein-Podolsky-Rosen Paradox:

- polarizations of pair of photons connected, no matter how far apart they travel



"*spukhafte Fernwirkung*" = spooky action at a distance

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(\,|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\,\Big)$$

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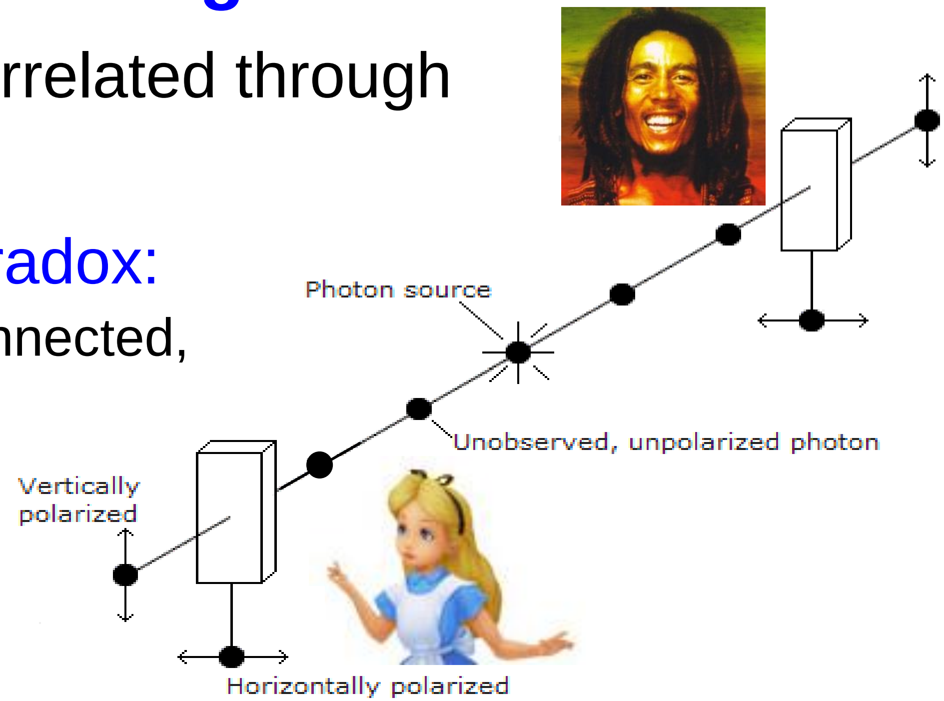

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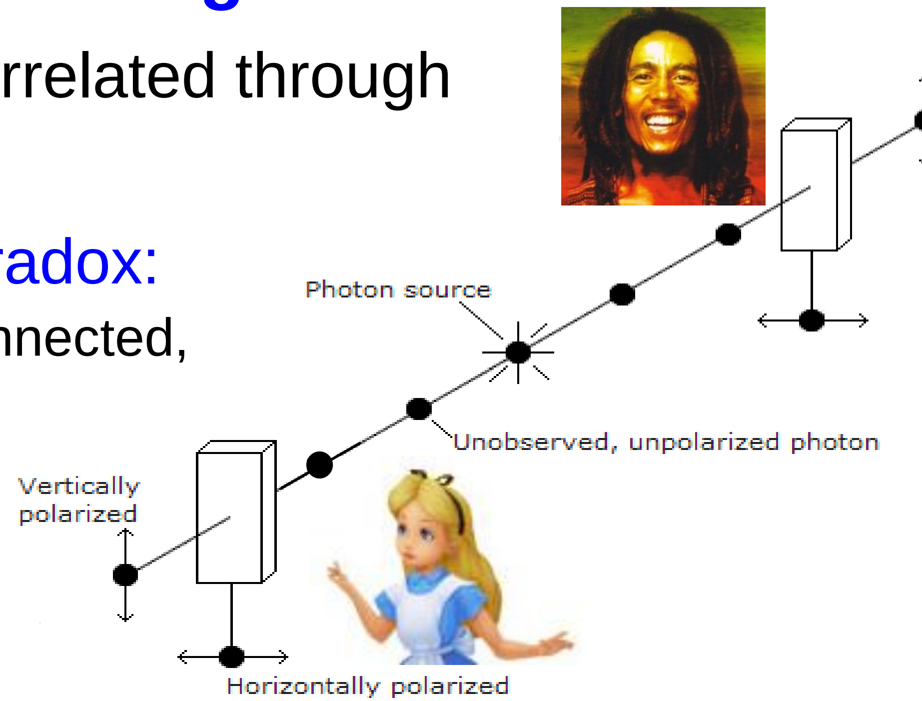
Quantum Information: entanglement becomes a resource for (ultra)fast computations and (ultra)secure communications

Condensed Matter: key to "exotic" phases and phenomena, e.g., quantum Hall fluids, unconventional superconductors, quantum spin fluids, ...

Quantum Entanglement



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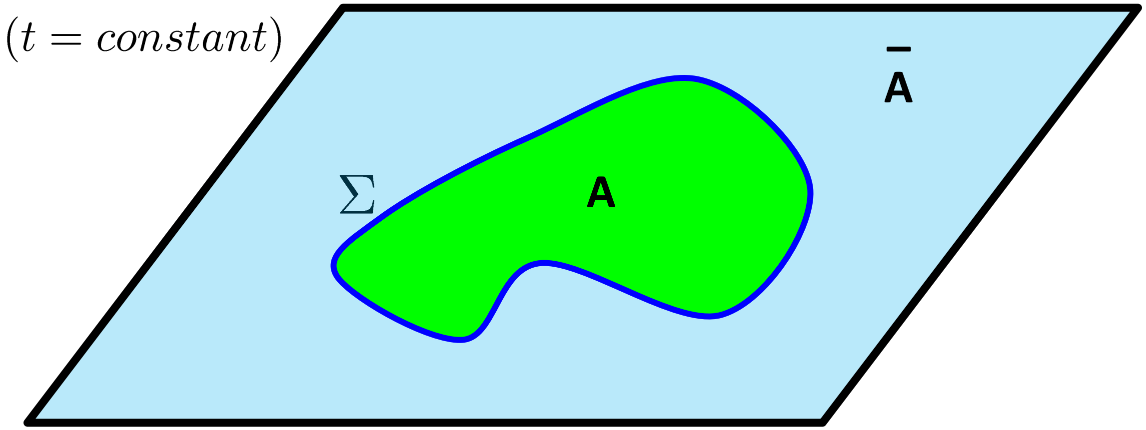
Quantum Fields & Quantum Gravity

Entanglement Entropy in QFT

- general diagnostic to give a quantitative measure of entanglement using entropy to detect correlations between two subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in “outside” region
- remaining dof are described by a density matrix ρ_A

→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



Holography: AdS/CFT correspondence

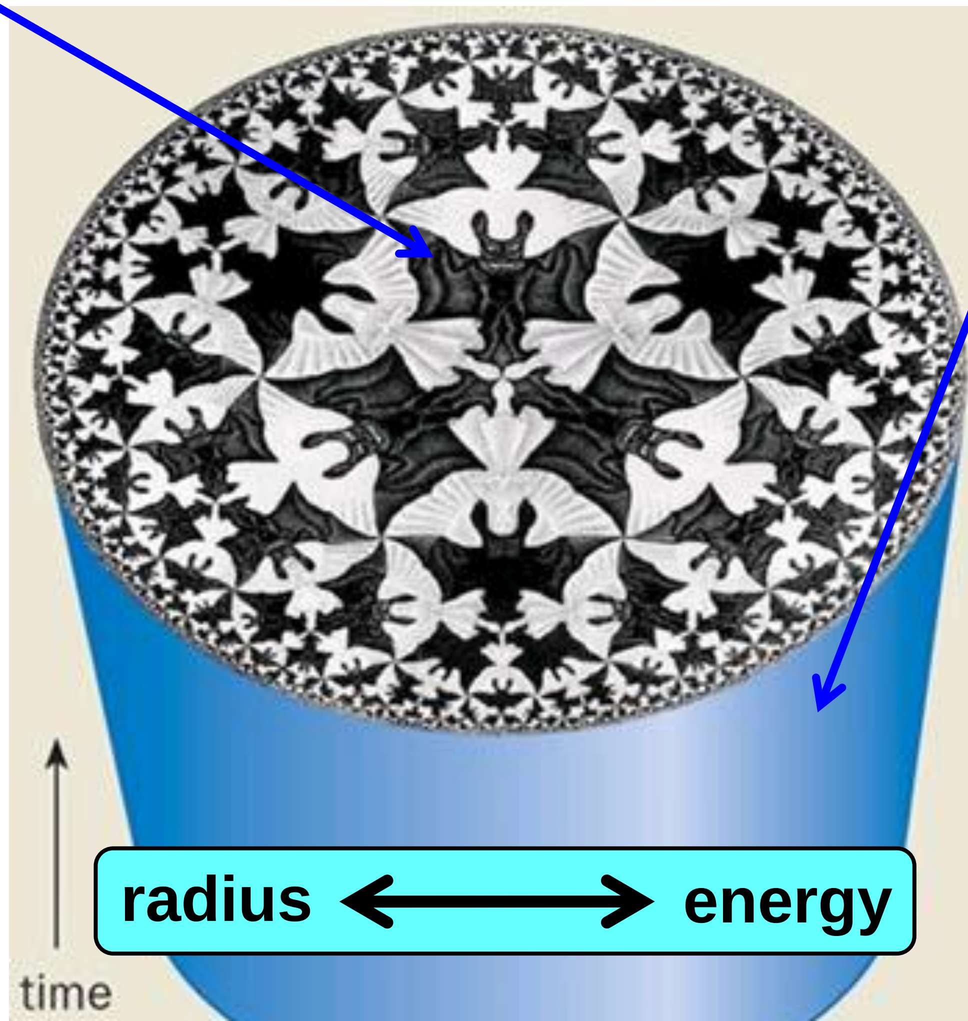
<u>Bulk</u>: gravity with negative Λ
in **d+1** dimensions

"holography"

<u>Boundary</u>: quantum field theory
without intrinsic scales
in **d** dimensions

anti-de Sitter
space

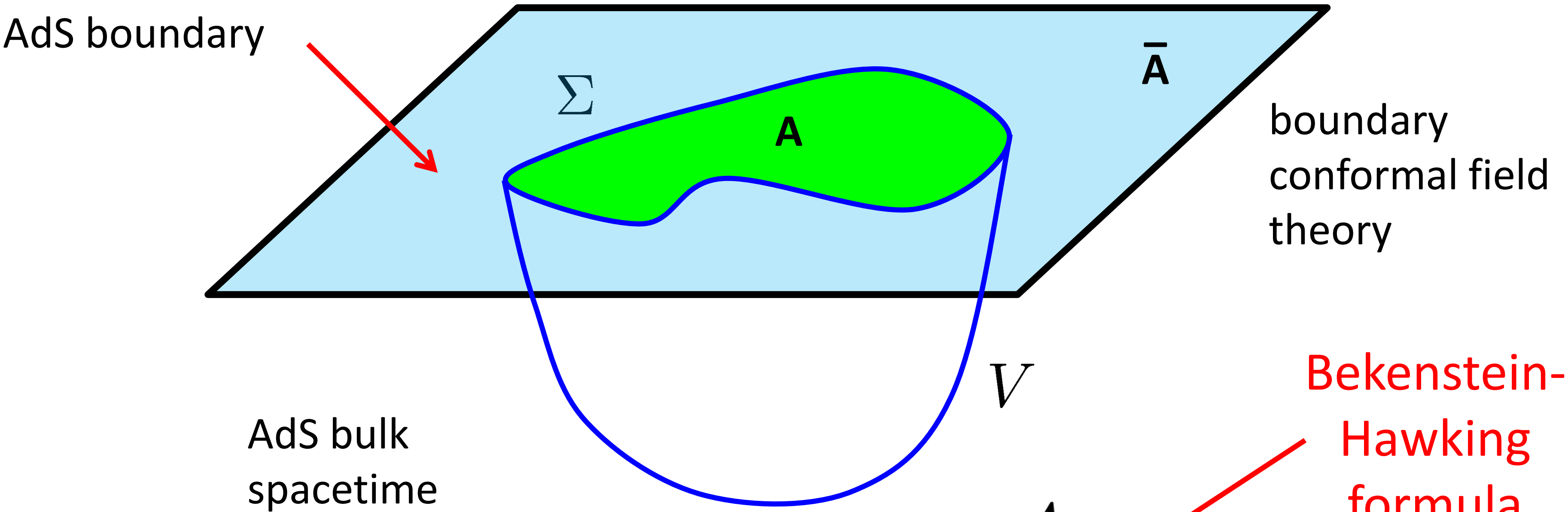
conformal
field theory



(Maldacena '97)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \underset{V \sim A}{\mathrm{ext}} \frac{A_V}{4G_N}$$

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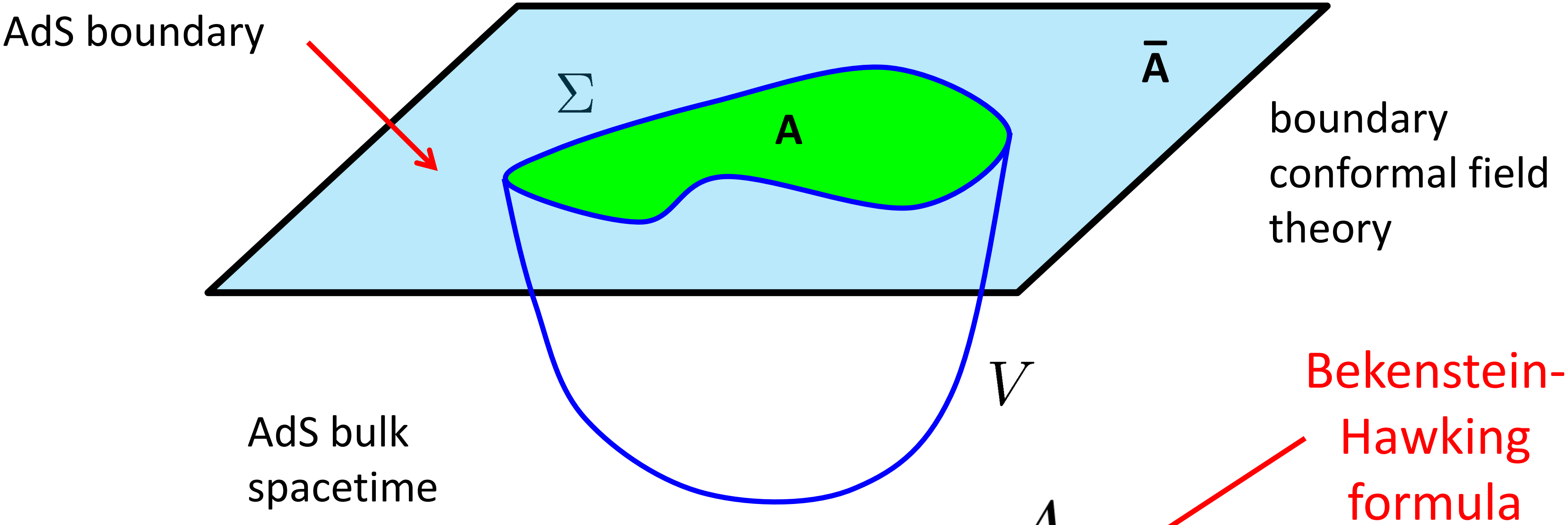

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• conjecture ⟶ many detailed consistency tests

(Ryu, Takayanagi, Hubeny, Rangamani, Headrick, Hung, Smolkin, RM, Faulkner, . . .)

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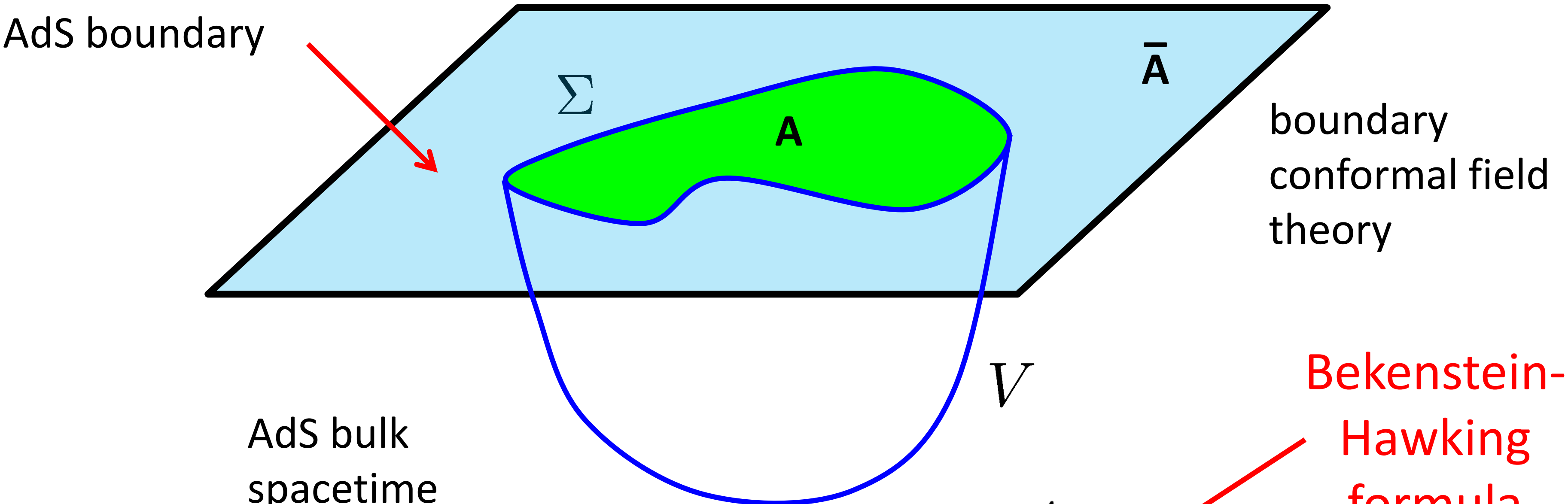

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- 2013 proof (for static geometries) (Maldacena & Lewkowycz)
- 2016 proof (for general geometries) (Dong, Lewkowycz & Rangamani)

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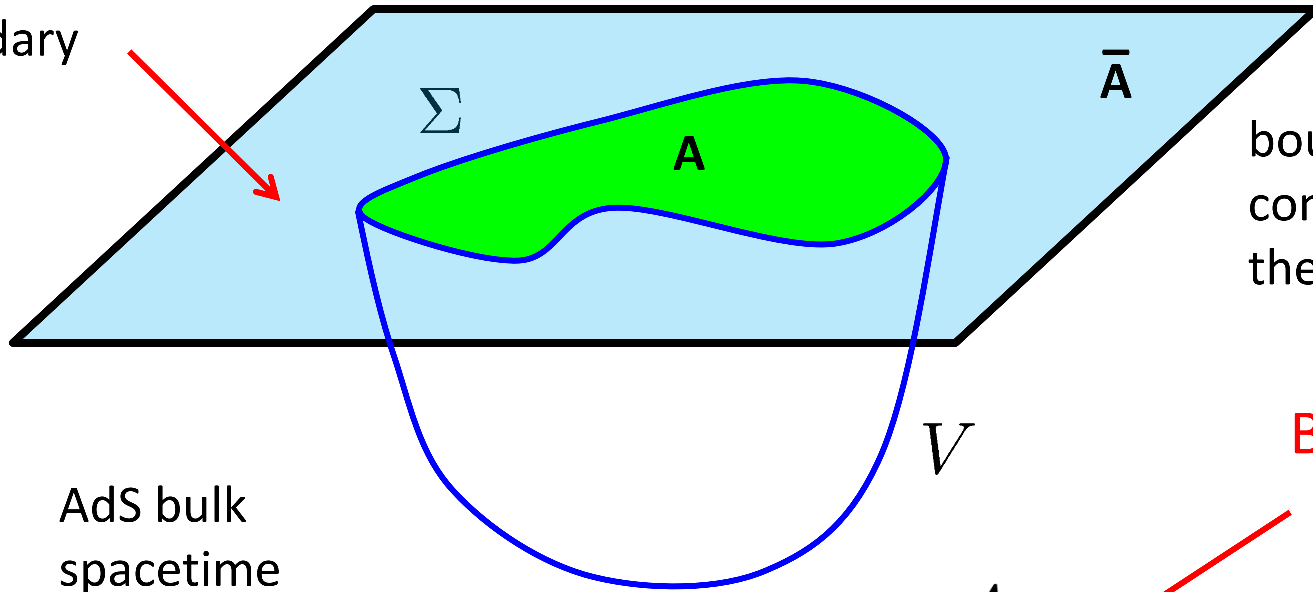

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- holographic EE: fruitful forum for bulk-boundary dialogue

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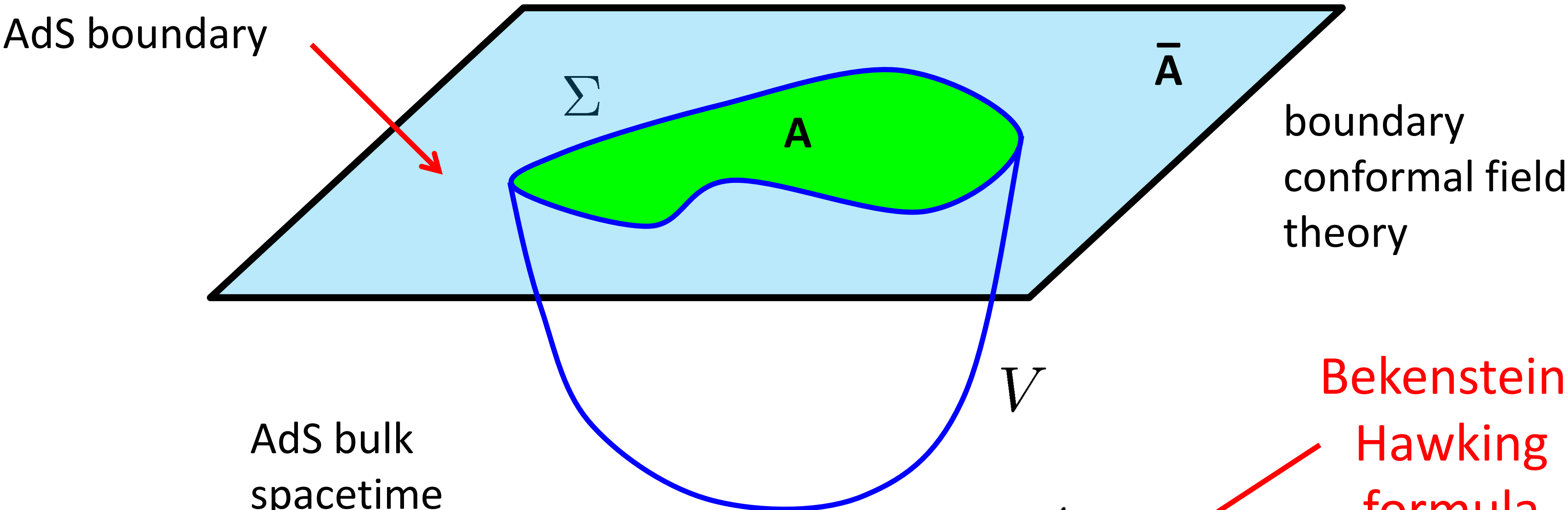


Bekenstein-Hawking formula

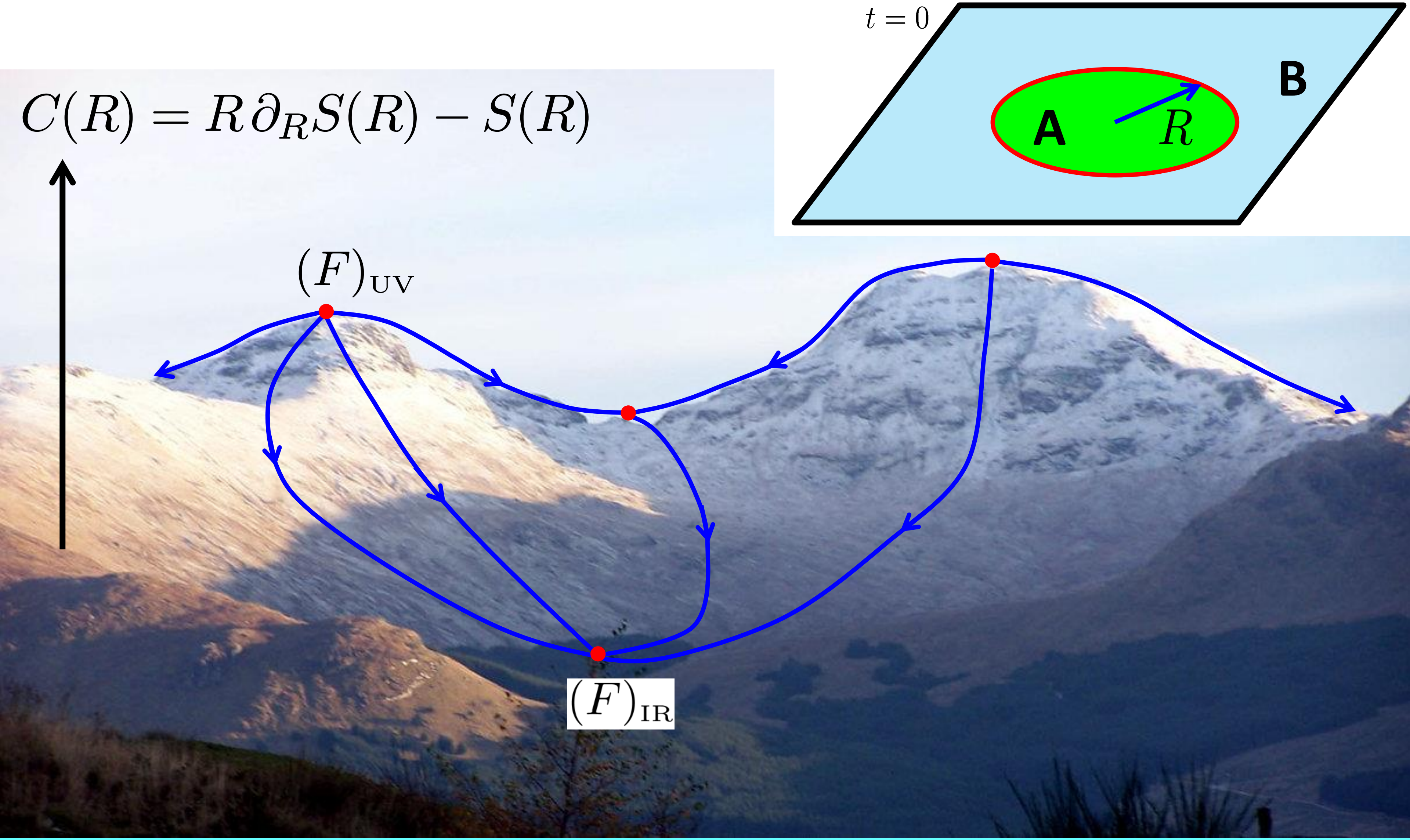
$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N}$$

- holographic EE teaches us lessons about QFTs, eg,

⟶ diagnostic in RG flows and c-theorms, eg, F-theorem (Sinha & RM, ……)

gravity/holography + EE → RG flows in (2+1)-dimensions

$$C(R) = R\,\partial_R S(R) - S(R)$$



F-theorem: $(F)_{UV} \;\geq\; (F)_{IR}$

(Ryu & Takayanagi `06)

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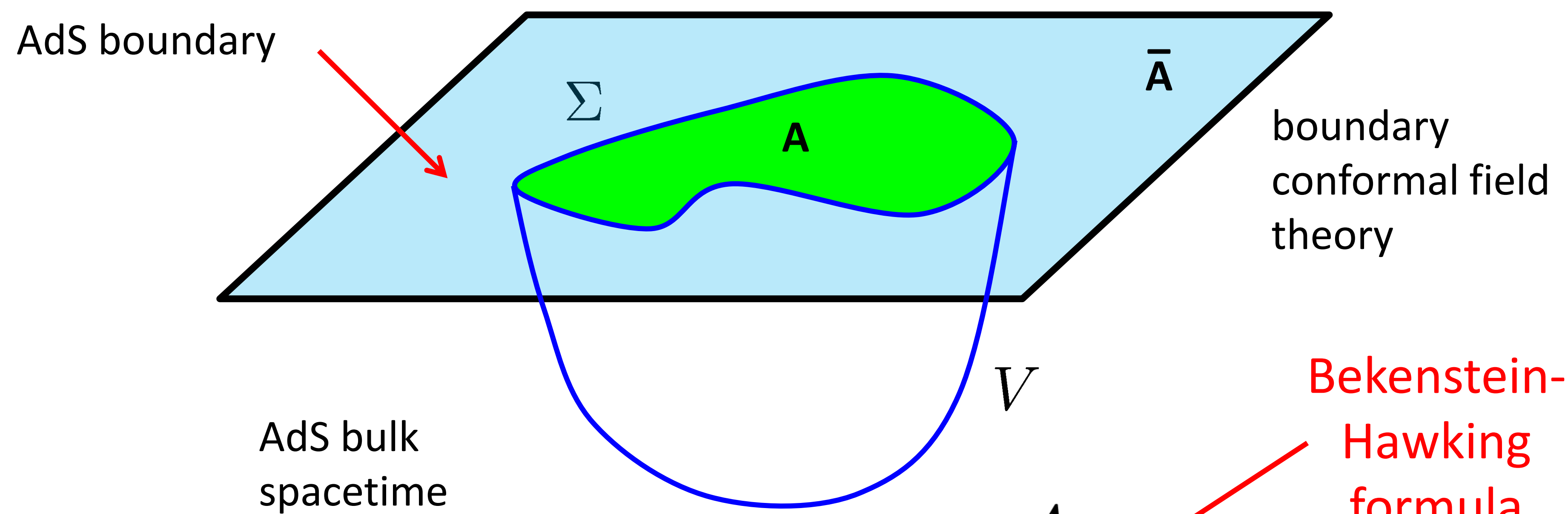

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  - → diagnostic in RG flows and c-theorms, eg, F-theorem (Sinha & RM, ……)
  - → geometric properties of entanglement entropy in QFT's (Mezei, Perlmutter, Lewkowycz, Bueno, RM, Witczak-Krempa, ….)
  - → diagnostic for quantum quenches/phase transitions (Lopez, Johnson, Balasubramanian, Bernamonti, Craps, Galli, ….)

(Ryu & Takayanagi `06)

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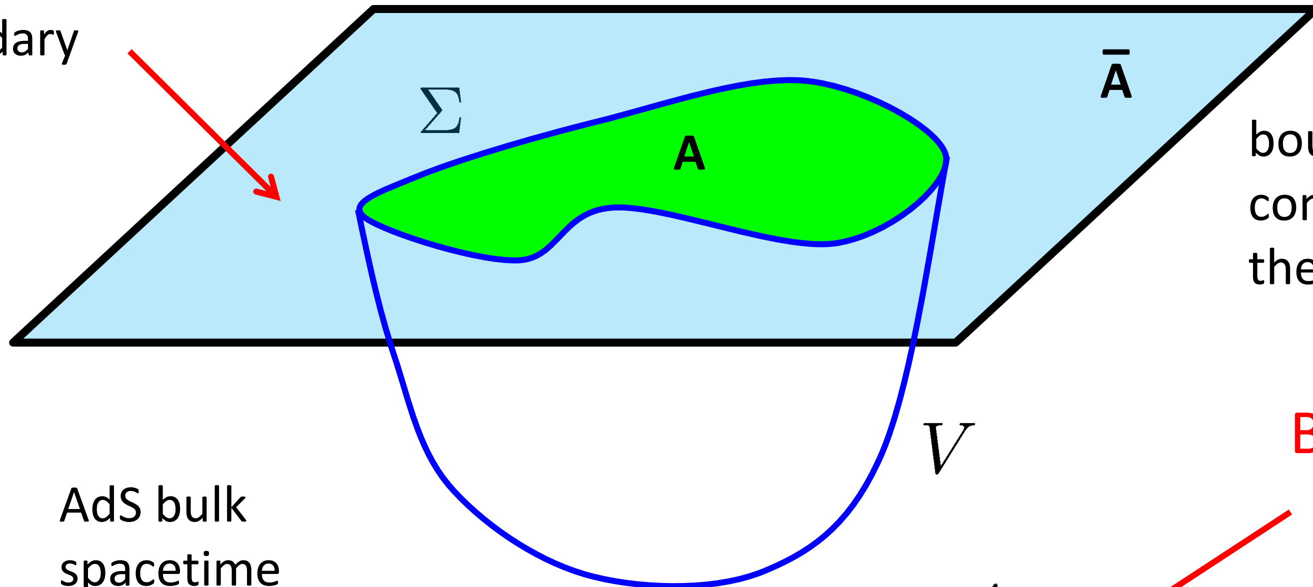

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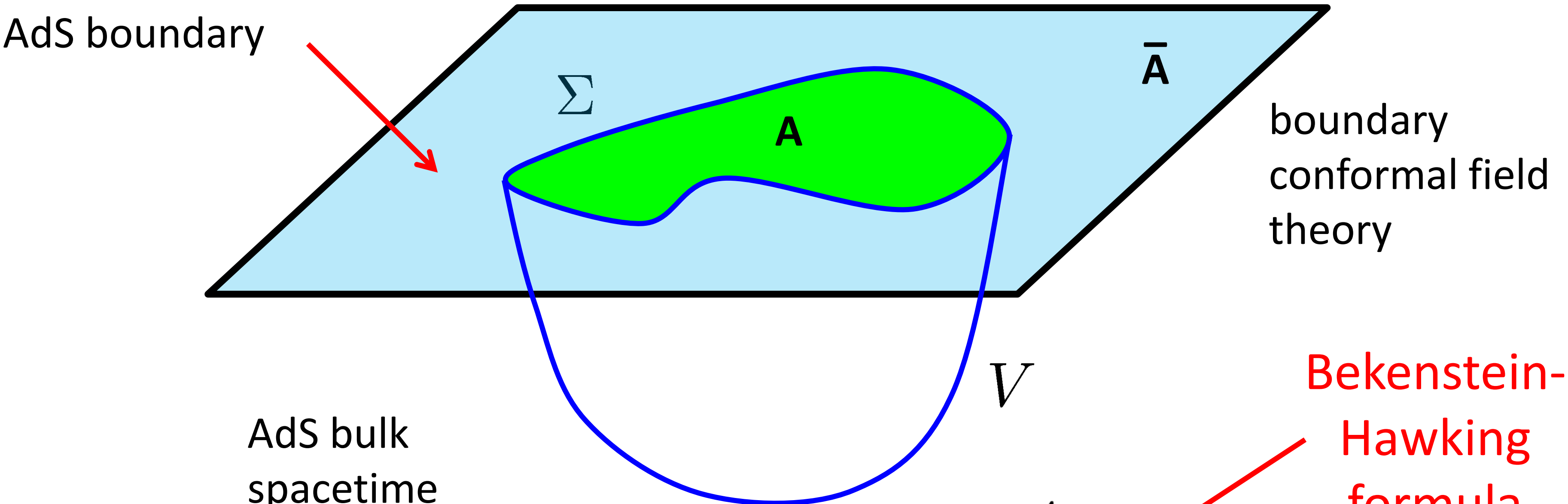

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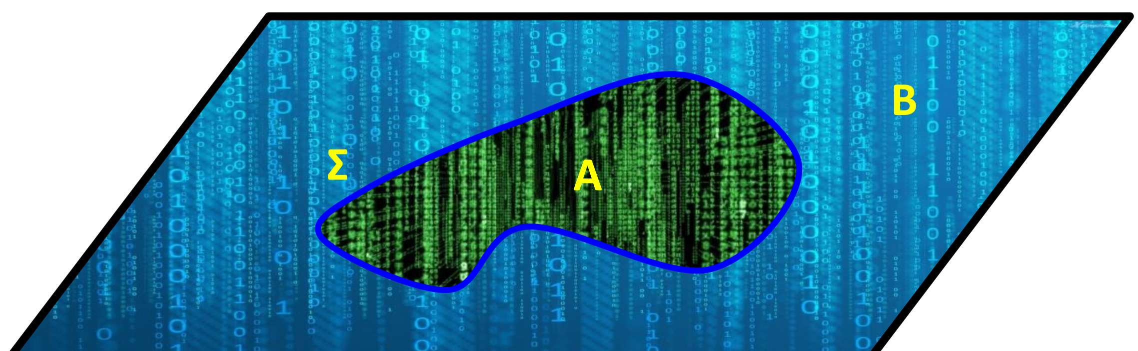
- → Bekenstein-Hawking formula: spacetime geometry encodes S_{BH}
- → black hole entropy is entanglement entropy (Sorkin, ….)
- → use BH formula for holographic entanglement entropy (Ryu & Takayanagi; ….)
- → connectivity of spacetime requires entanglement (van Raamsdonk)
- → spacetime entanglement conjecture (Bianchi & RM)
- → AdS spacetime as a tensor network (MERA) (Swingle, Vidal, ….)
- → "ER = EPR" conjecture (Maldacena & Susskind)
- → hole-ographic spacetime (Balasubramanian, Chowdhury, Czech, de Boer & Heller; RM, Rao & Sugishita; Czech, Dong & Sully; ….)



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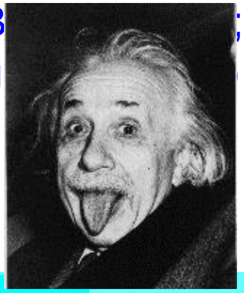

spacetime provides both the stage for physical phenomena and the agent which manifests gravitational dynamics

Gravitational Dynamics from Entanglement:

(Lashkari, McDermott & Van Raamsdonk; Swingle & Van Raamsdonk; Faulkner, Guica, Hartman, RM & Van Raamsdonk)

- entanglement entropy: $S(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A)$
- make a small perturbation of state: $\tilde{\rho} \;=\; \rho_A + \delta\rho$

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(Blanco, Casini, Hung & RM)

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"1st law" of entanglement entropy

- this is **the** 1st law for thermal state: $\rho_A = \exp(-H/T)$

"1st law" of entanglement entropy: $\delta S_A = \delta \langle H_A \rangle$

- generally H_A "**nonlocal mess**" and flow is **not geometric**

$$H_A = \int d^{d-1}x \, \gamma_1^{\mu\nu}(x) \, T_{\mu\nu} + \int d^{d-1}x \int d^{d-1}y \, \gamma_2^{\mu\nu;\rho\sigma}(x,y) \, T_{\mu\nu} T_{\rho\sigma} + \cdots$$

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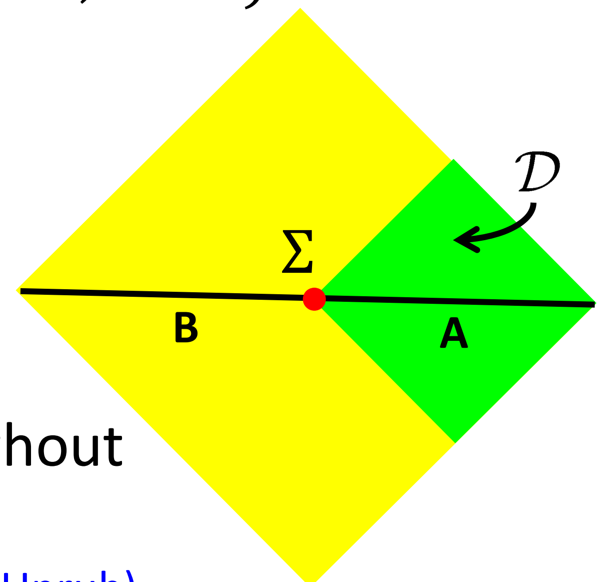
→ hence usefulness of first law is very limited, in general

- famous exception: **Rindler wedge**
- any QFT in Minkowski vacuum; choose $\Sigma = (\mathrm{x} = 0, \mathrm{t} = 0)$

$$H_A = 2\pi K \quad \longleftarrow \text{boost generator}$$

$$= 2\pi \int_{A(x>0)} d^{d-2}y\, dx\, [\, x\, T_{tt}\,] + c'$$



- by causality, ρ_A and H_A describe physics throughout domain of dependence $\mathcal{D}$

(Bisognano & Wichmann; Unruh)

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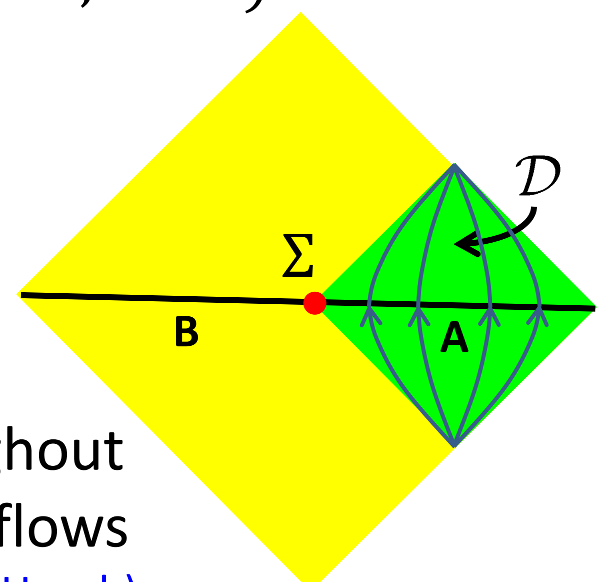

- by causality, ρ_A and H_A describe physics throughout domain of dependence $\mathcal{D}$; eg, generate boost flows

(Bisognano & Wichmann; Unruh)

"1st law" of entanglement entropy: $\delta S_A = \delta\langle H_A\rangle$

• **another exception**: CFT in vacuum of d-dim. flat space and entangling surface which is S^{d-2} with radius R

$$H_A = 2\pi \int_A d^{d-1}y \; \frac{R^2 - |\vec{y}|^2}{2R} T_{tt}(\vec{y}) + c'$$

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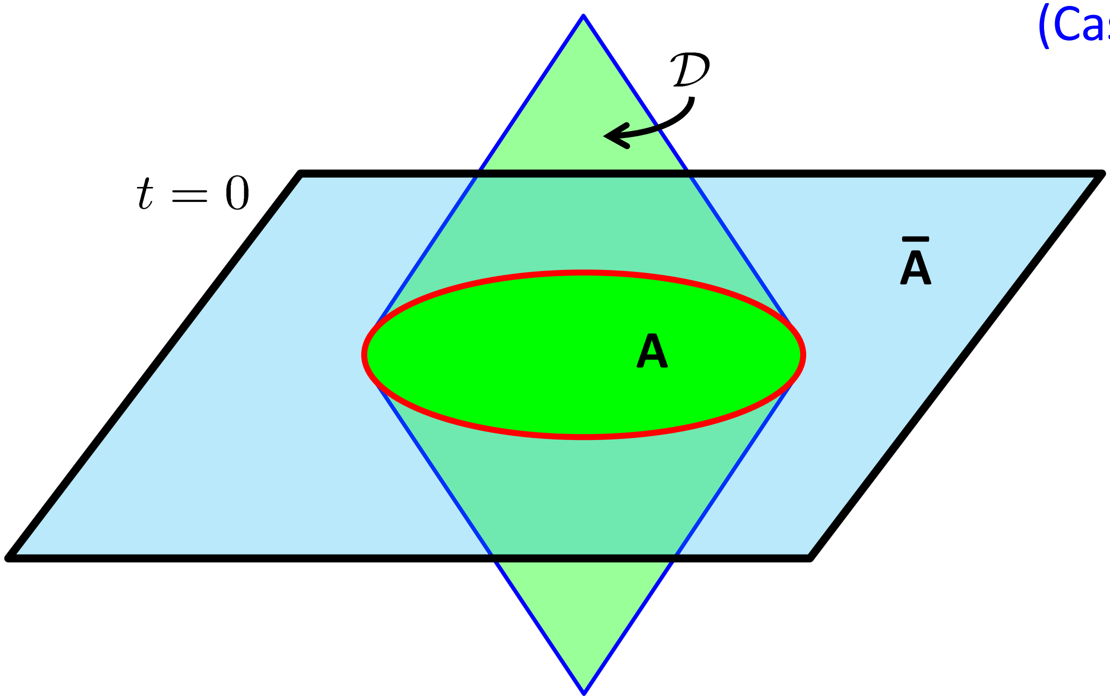

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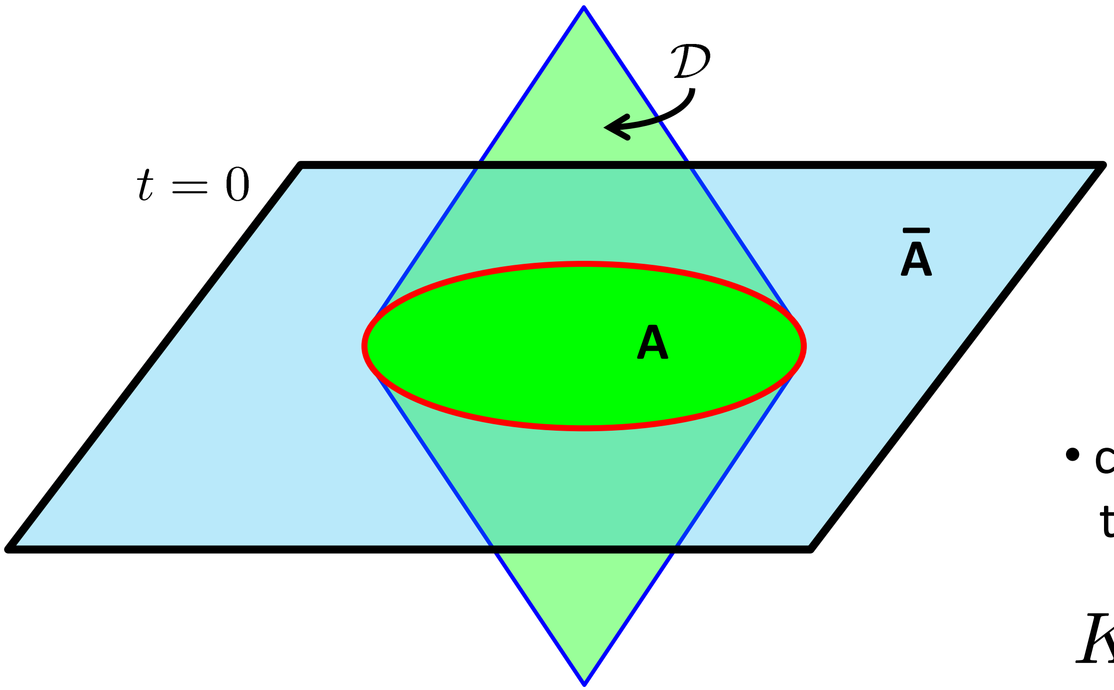

- construct with conformal transformation:

$$K = 2\pi \int_A d\Sigma^\mu \, T_{\mu\nu} \, K^\nu$$

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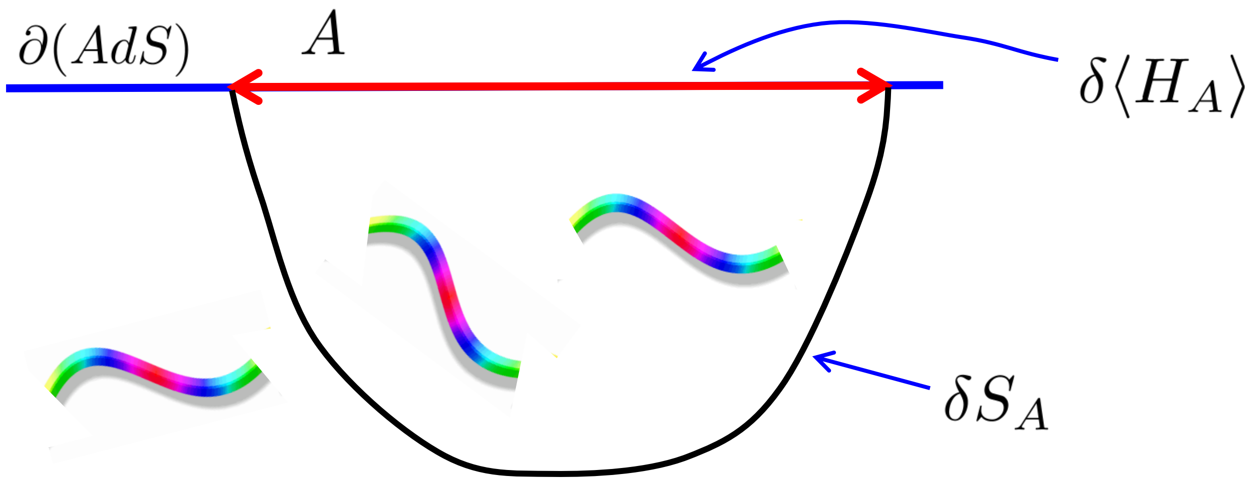

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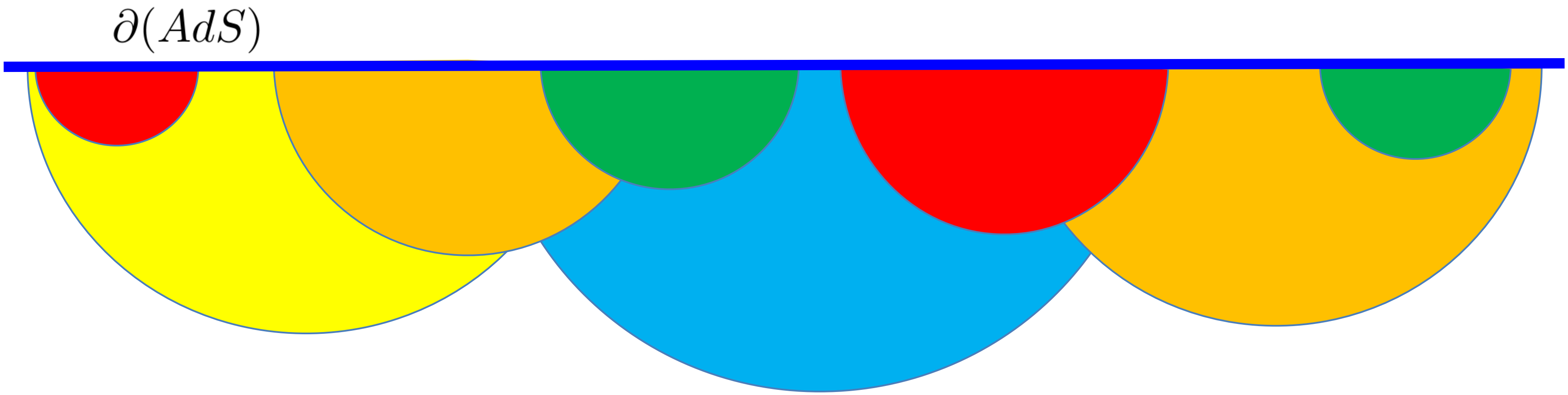

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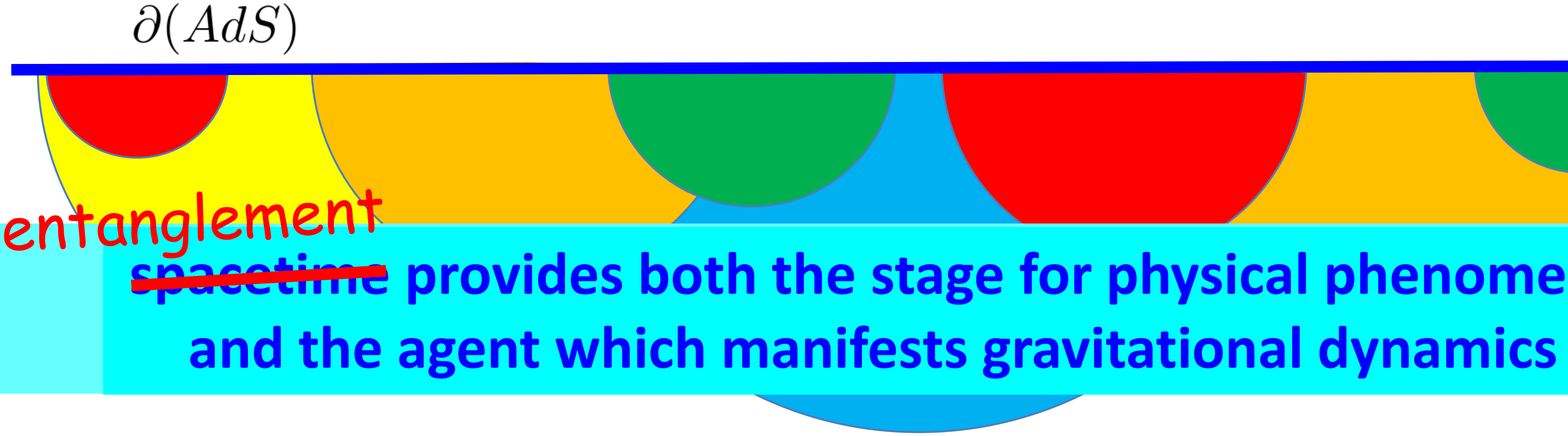

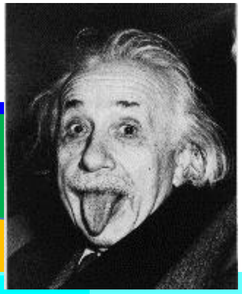

entanglement ~~spacetime~~ **provides both the stage for physical phenomena and the agent which manifests gravitational dynamics**

- apply 1st law for spheres of all sizes, positions and in all frames:

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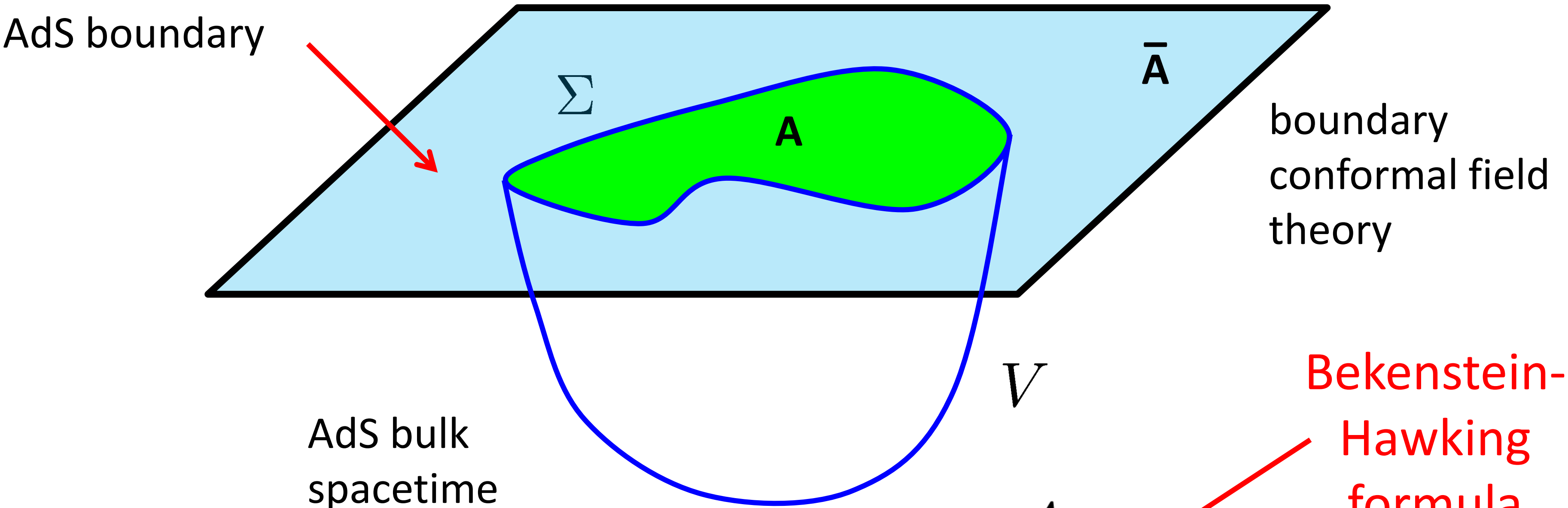
(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \underset{V \sim A}{\text{ext}} \; \frac{A_V}{4G_N}$$

- holographic EE teaches us lessons about (quantum) gravity, eg,

→ BH formula applies beyond black holes/horizons

→ connectivity of spacetime requires entanglement (van Raamsdonk)

Spacetime Geometry = Entanglement

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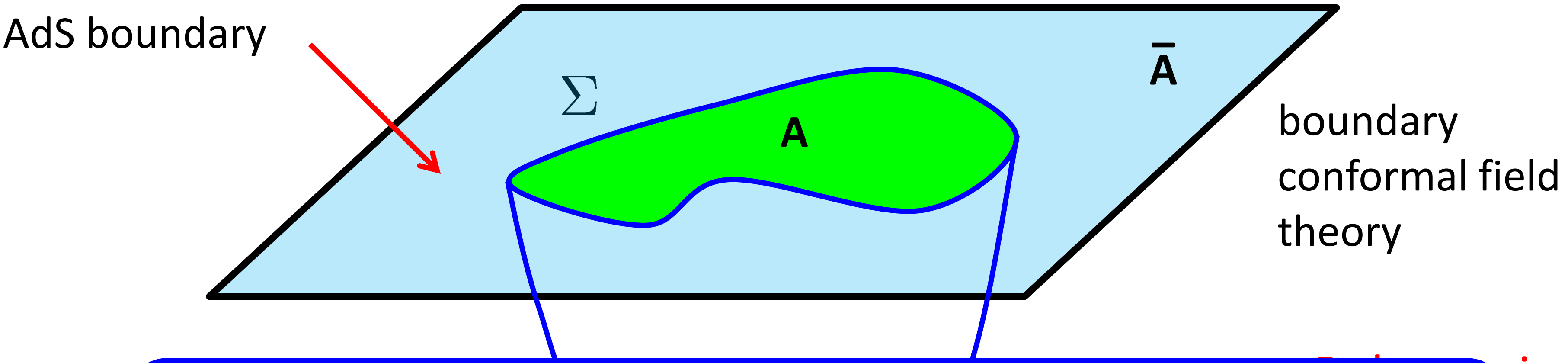

Susskind: Entanglement is not enough!

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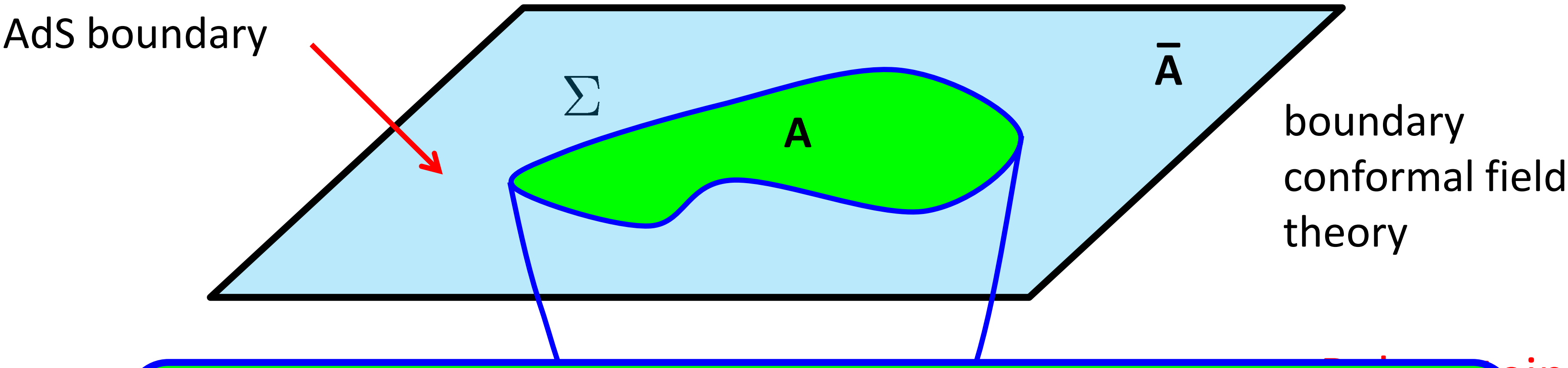

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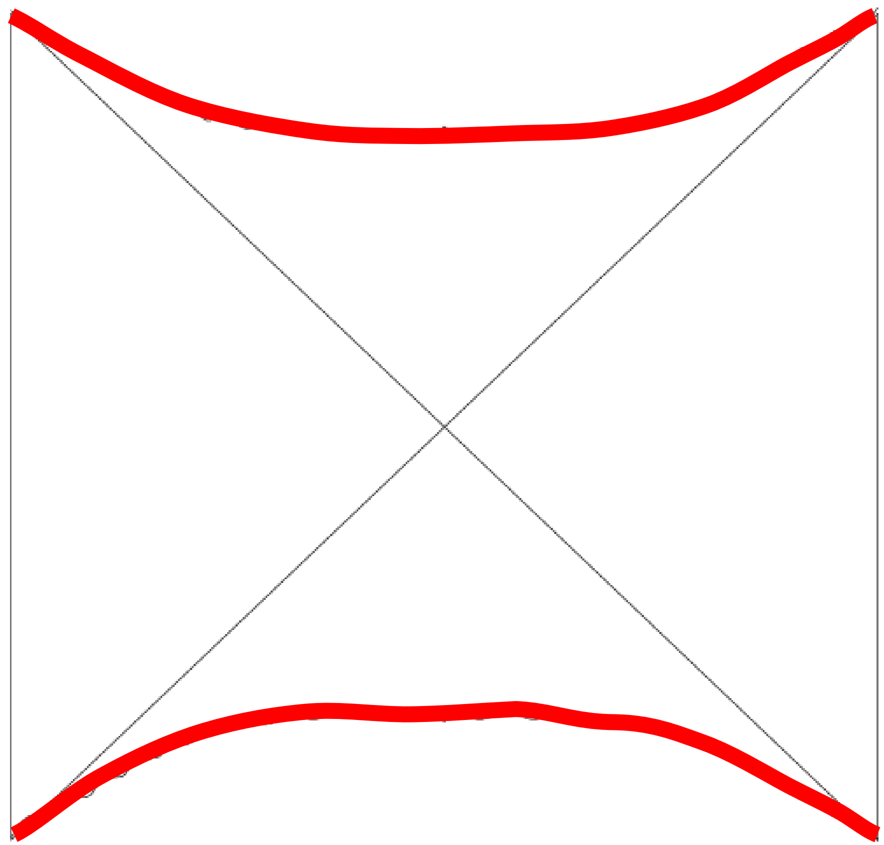

$$ds^2 = -\left(\frac{r^2}{L^2} + 1 - \frac{\mu}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1 - \frac{\mu}{r^{d-2}}} + r^2 d\;{}^2_{d-1}$$

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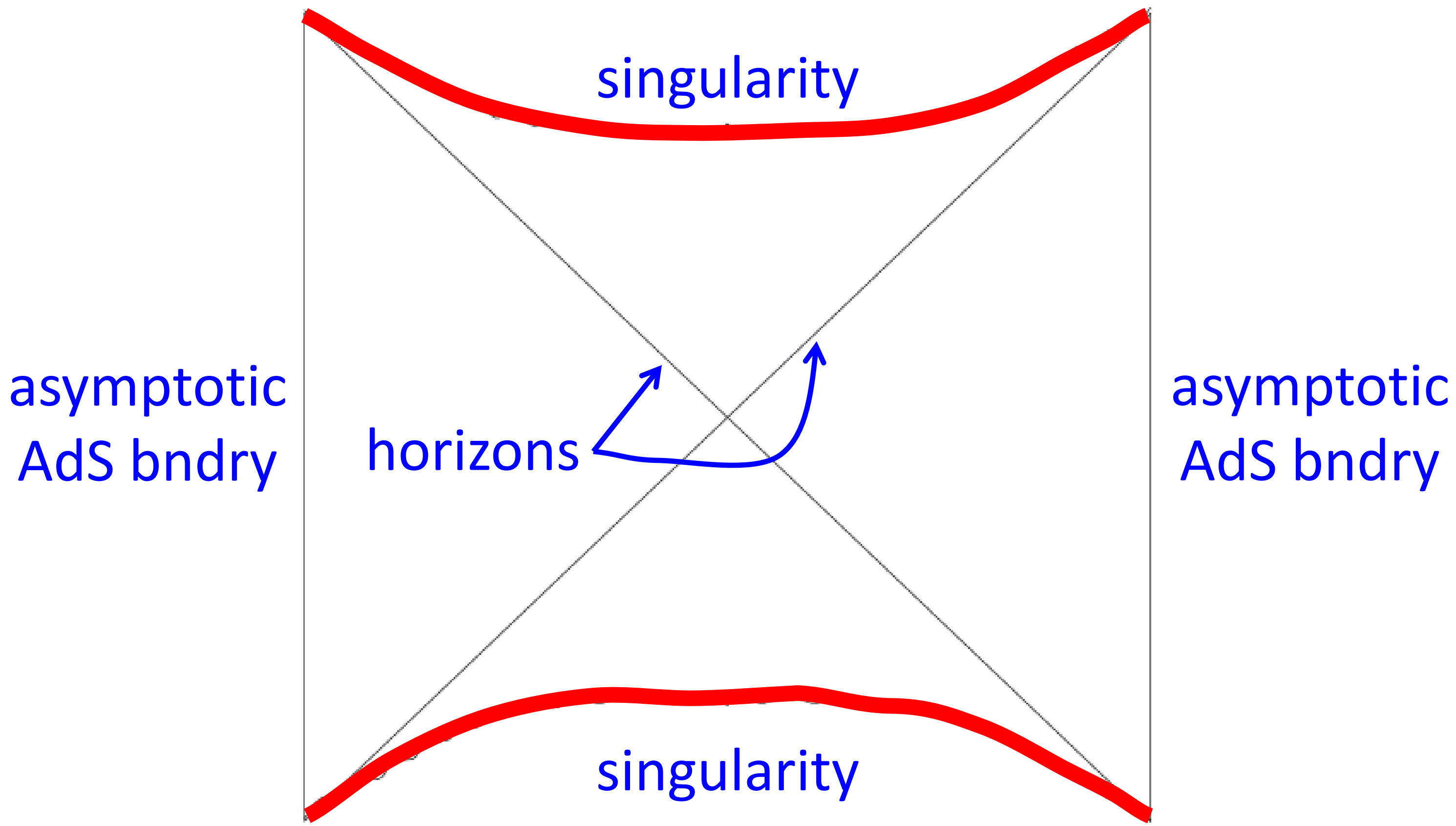

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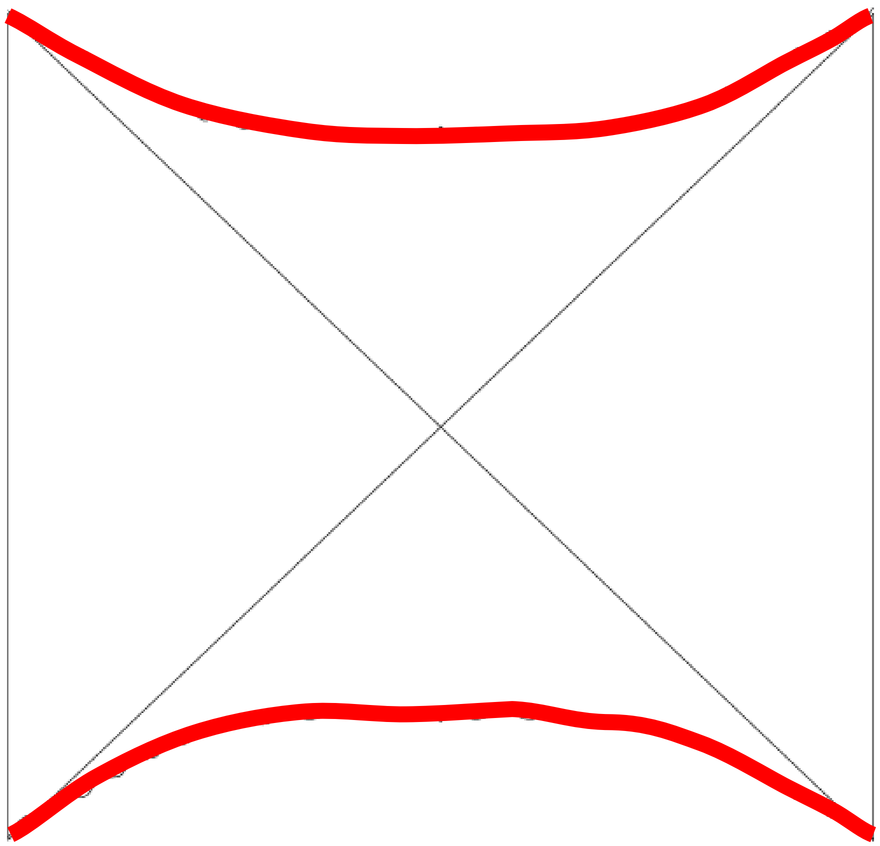

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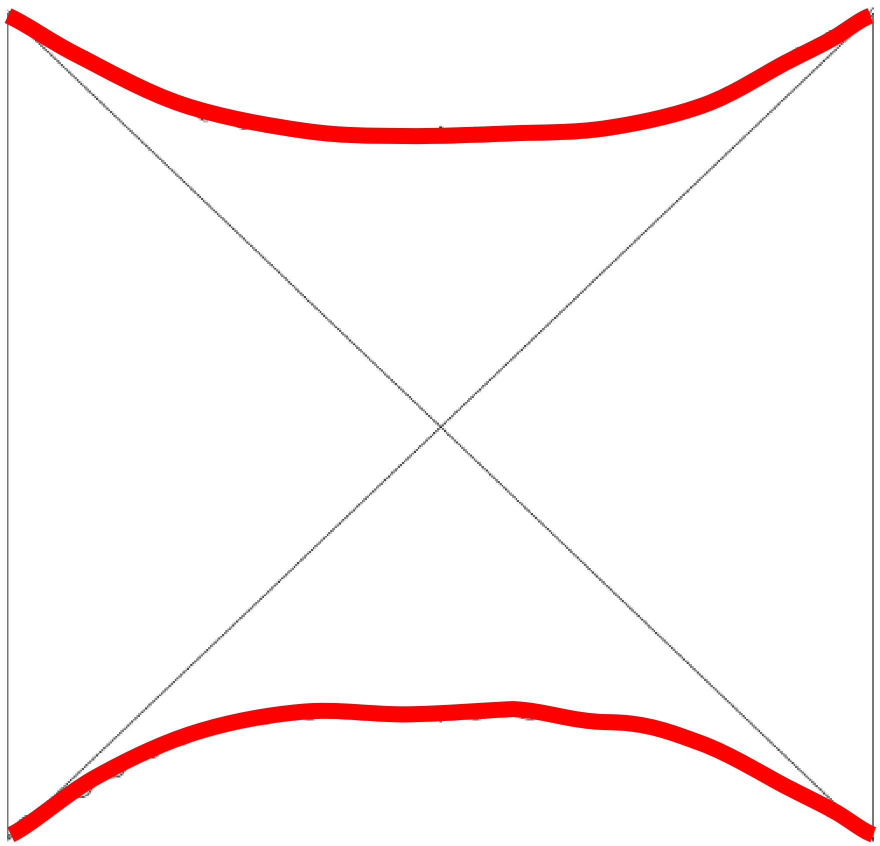

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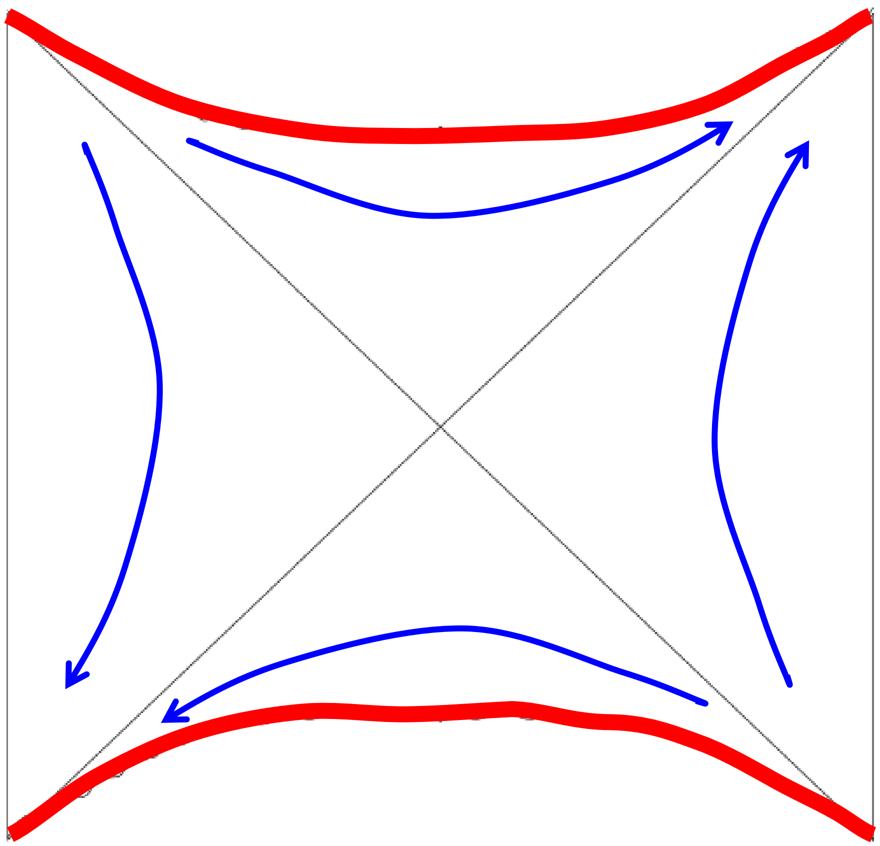

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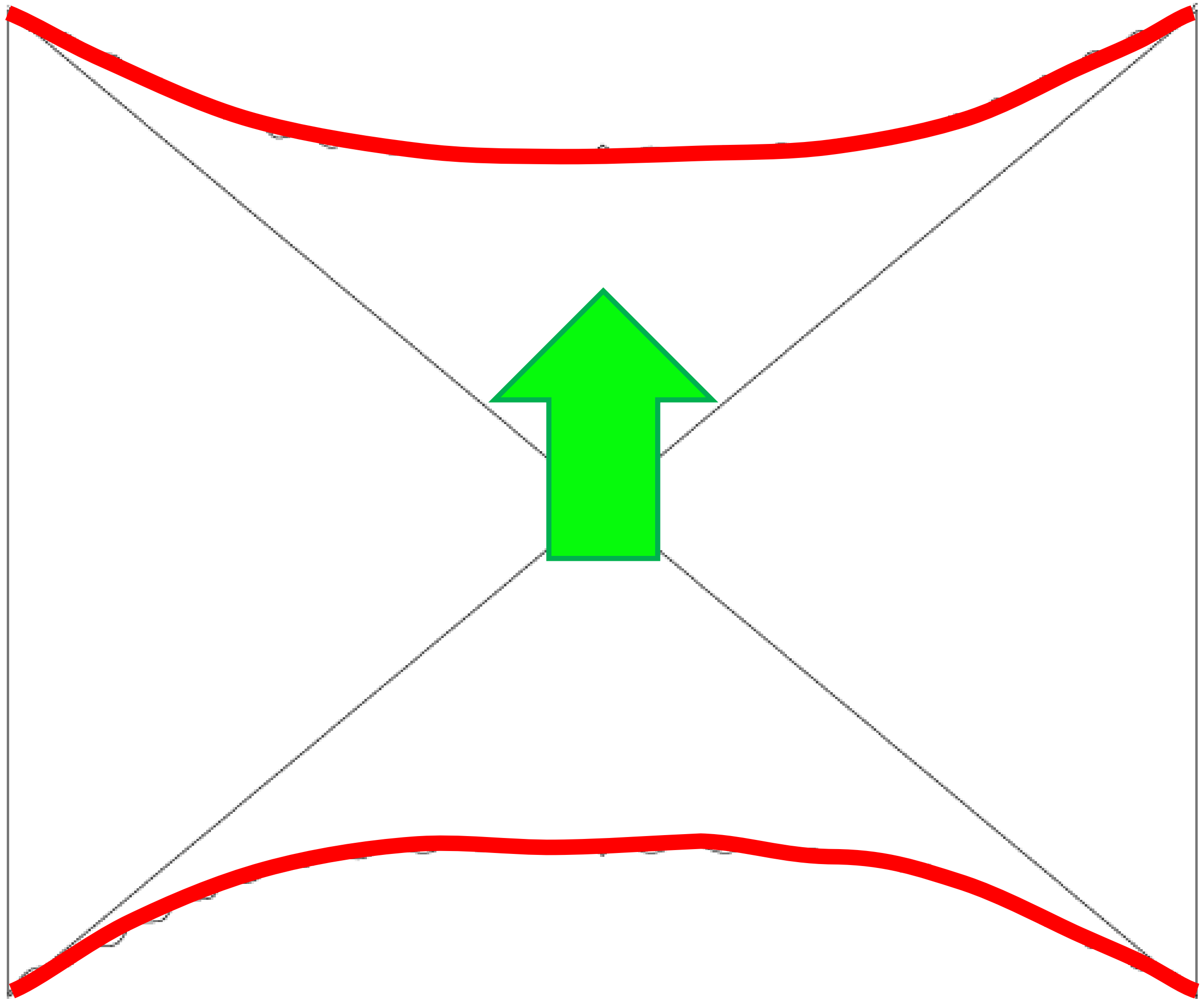

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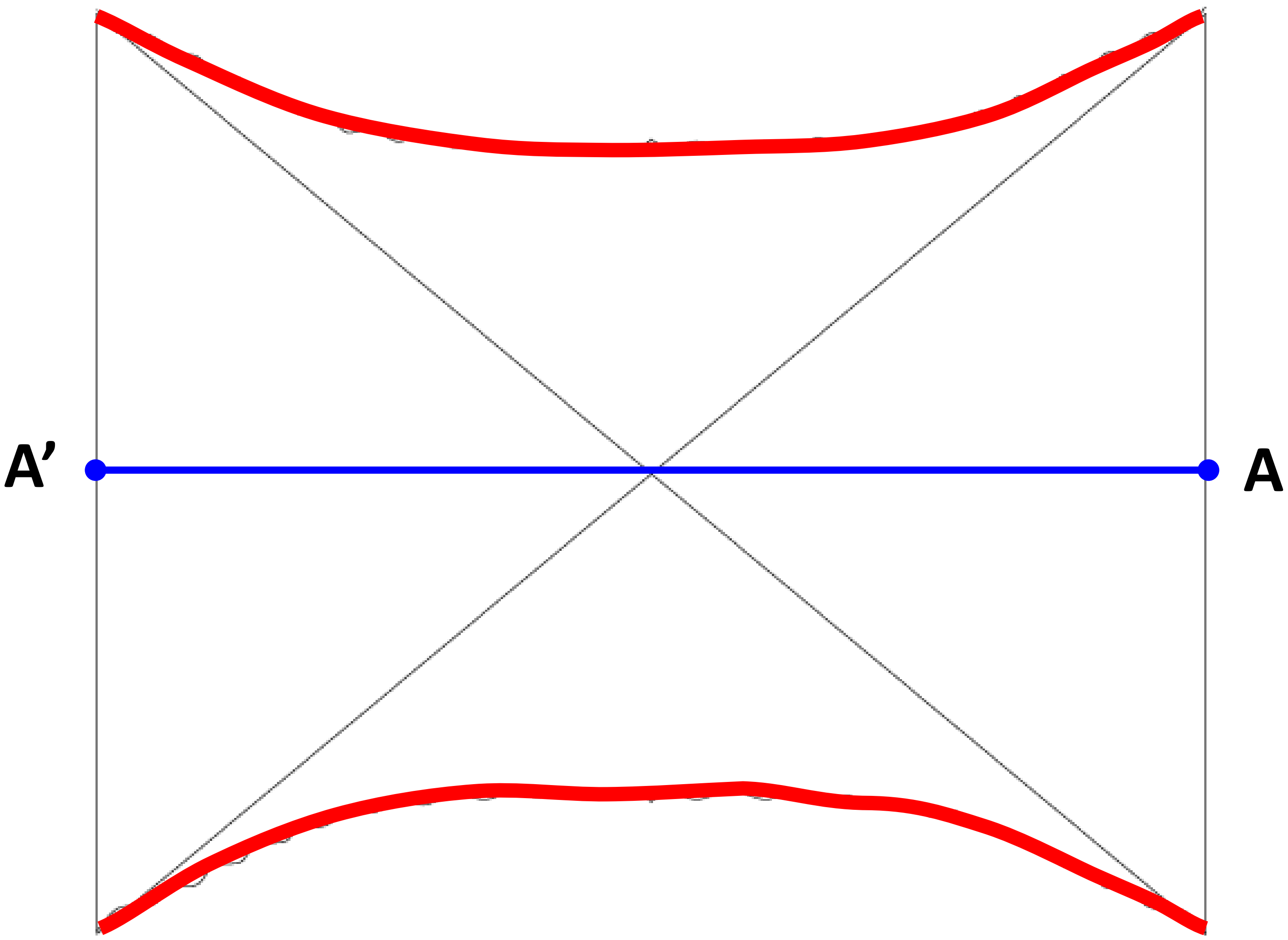

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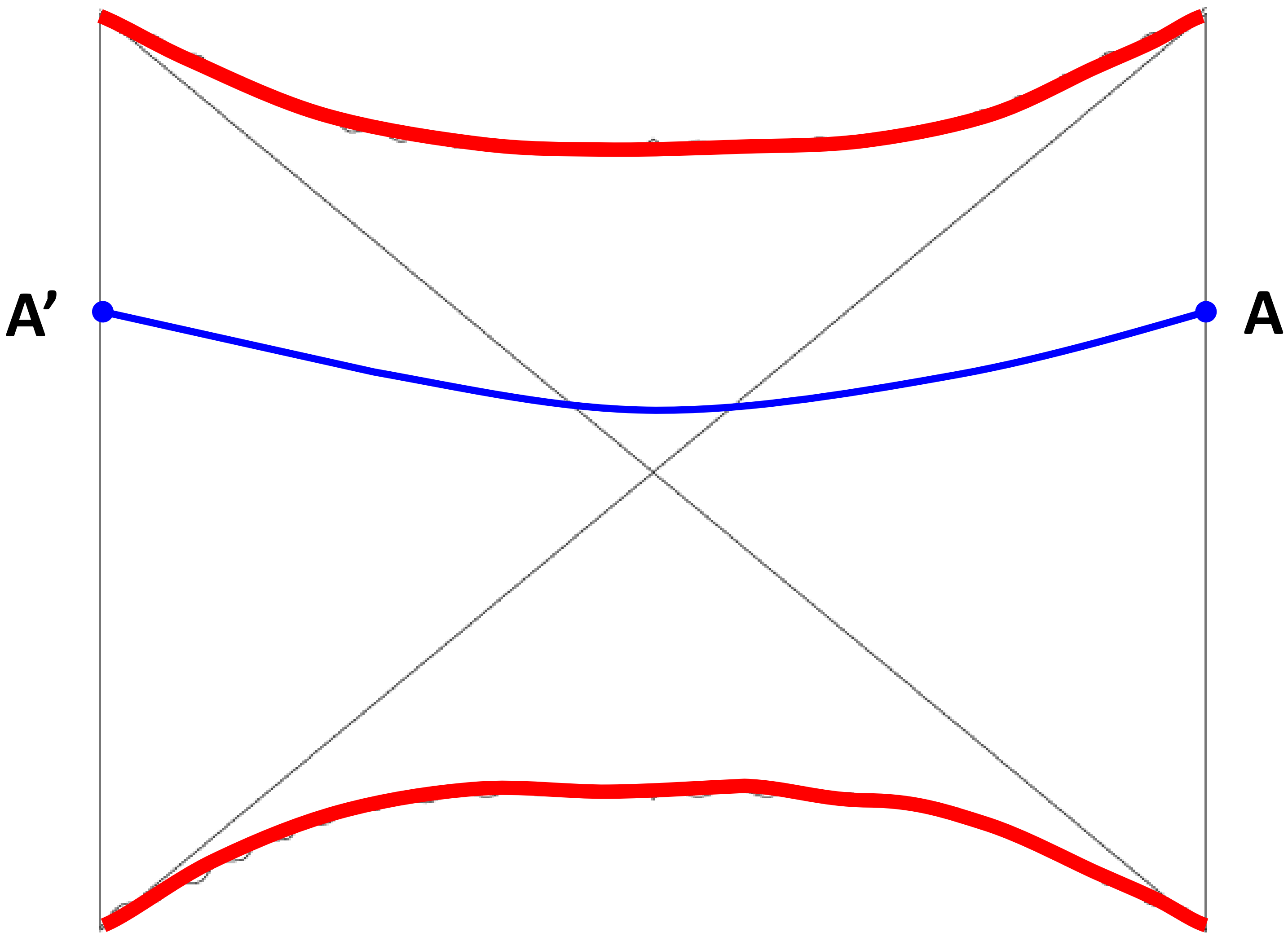

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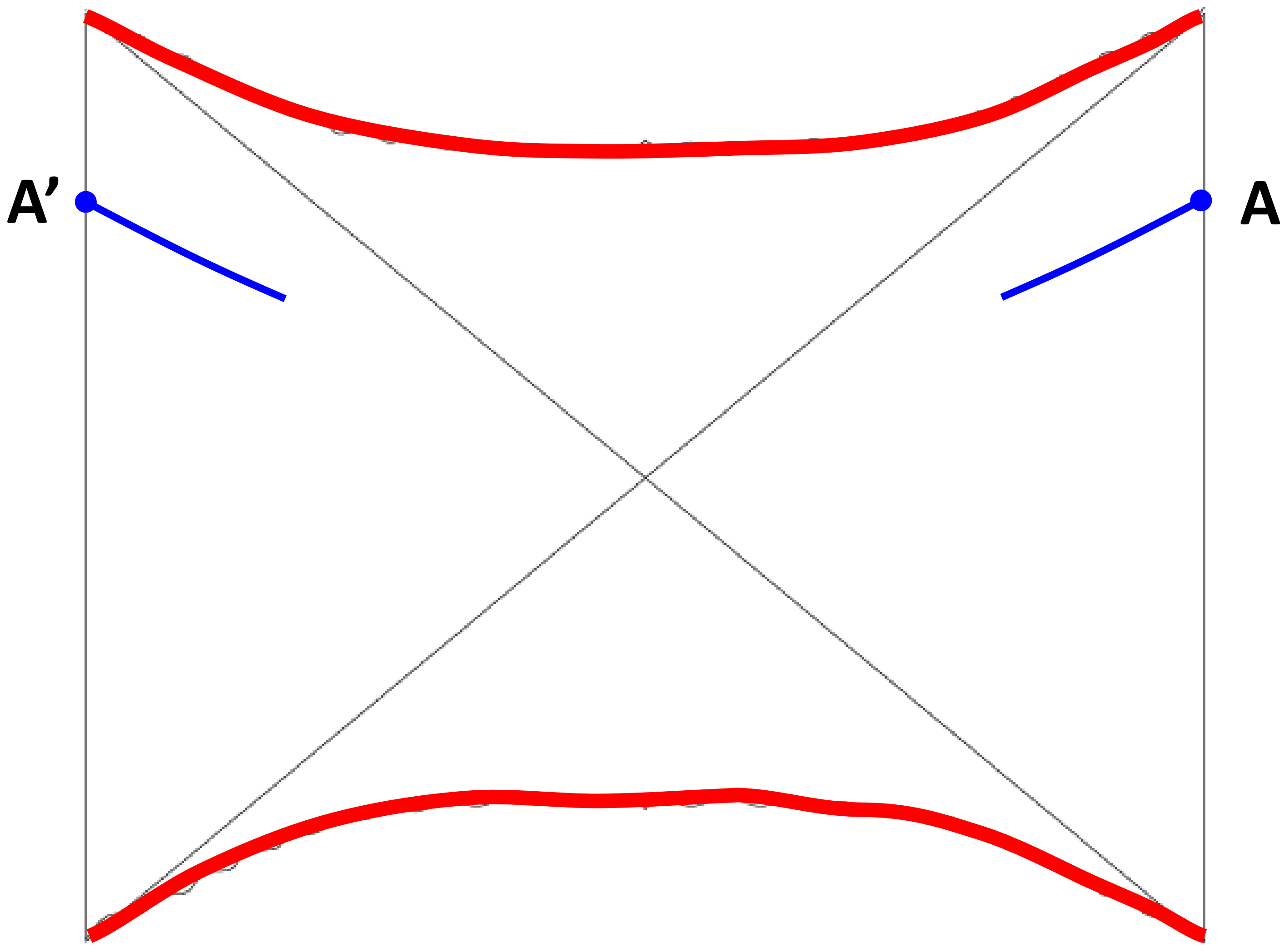

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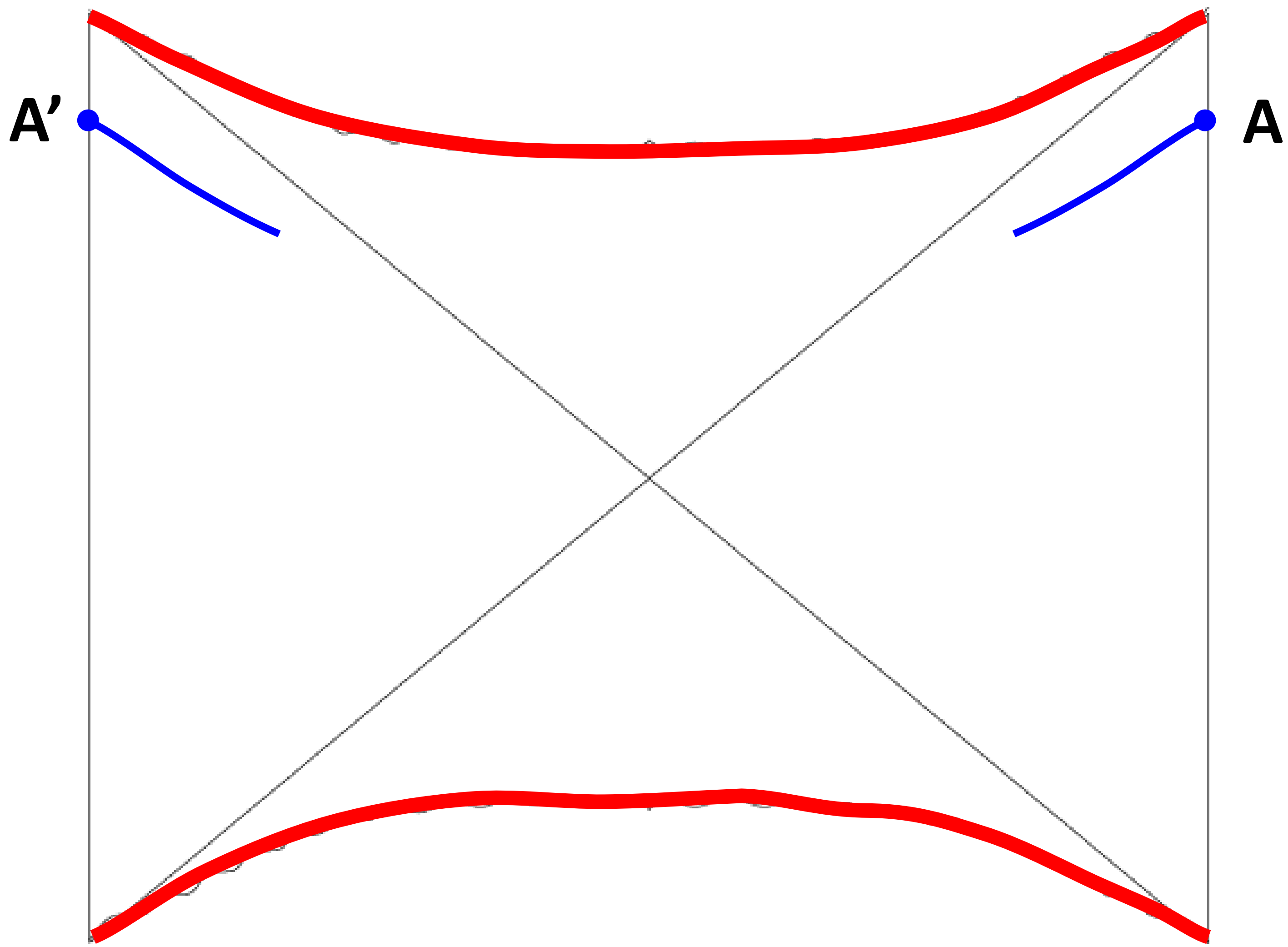

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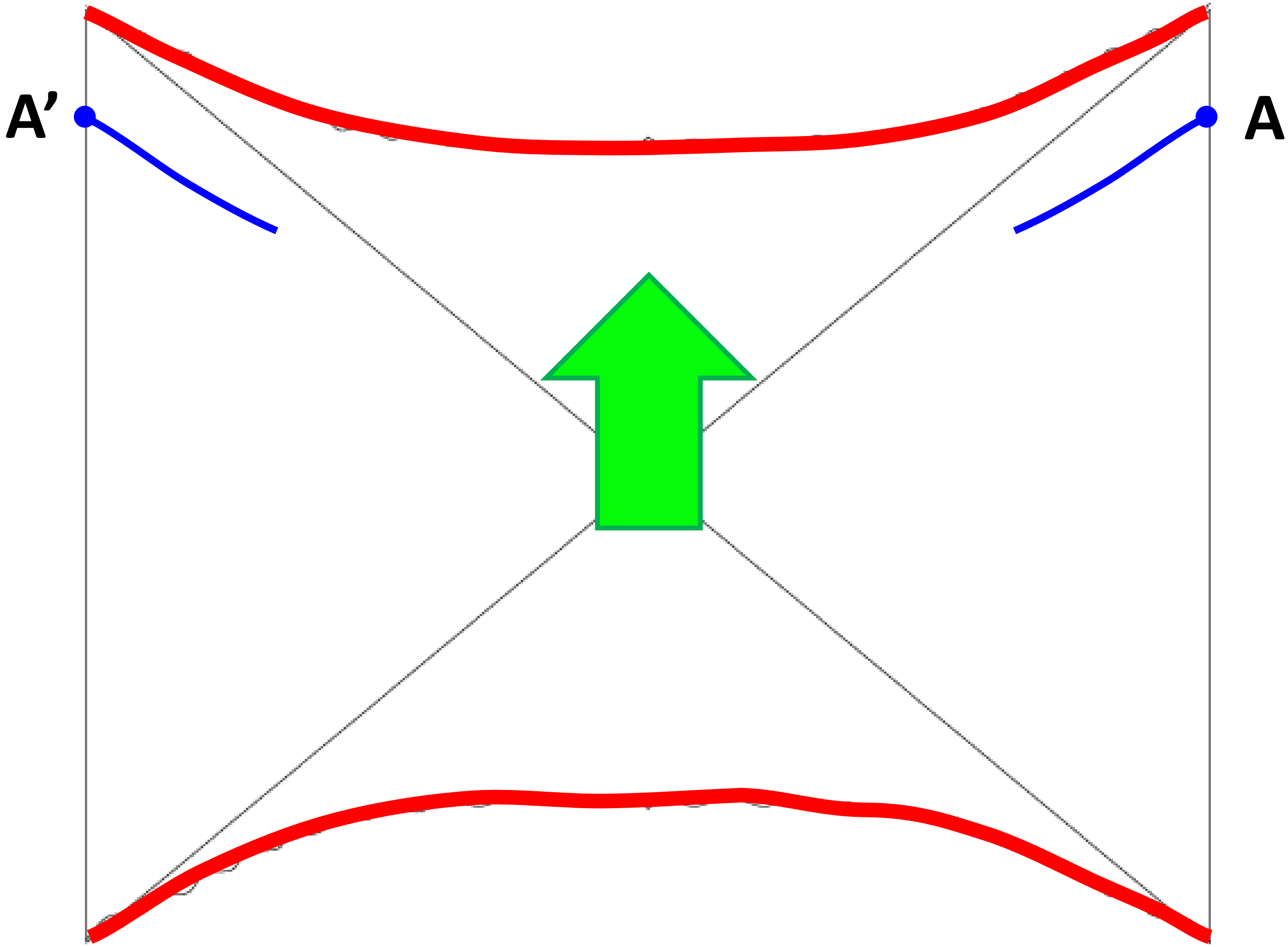

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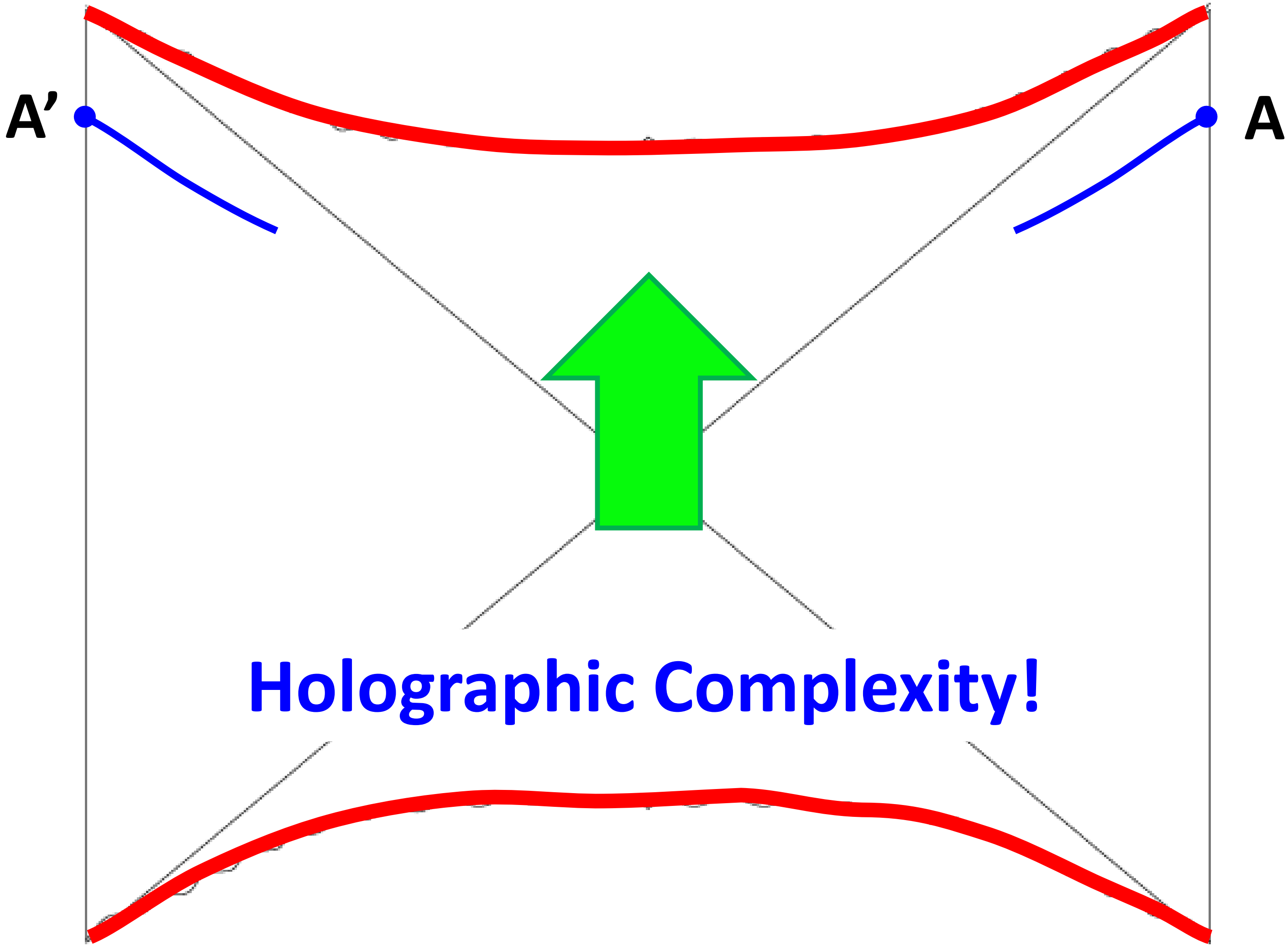

Complexity?

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- would like a new probe "sensitive to phases"

$$|\mathrm{TFD}\rangle \simeq \sum_\alpha e^{-E_\alpha/(2T) - iE_\alpha(t_L + t_R)} |E_\alpha\rangle_L |E_\alpha\rangle_R$$

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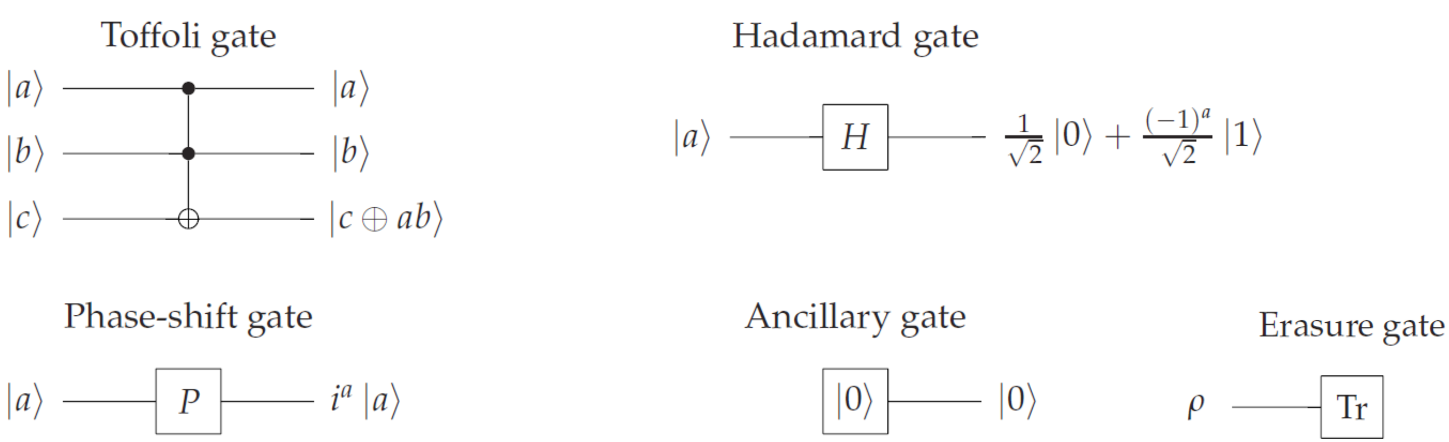

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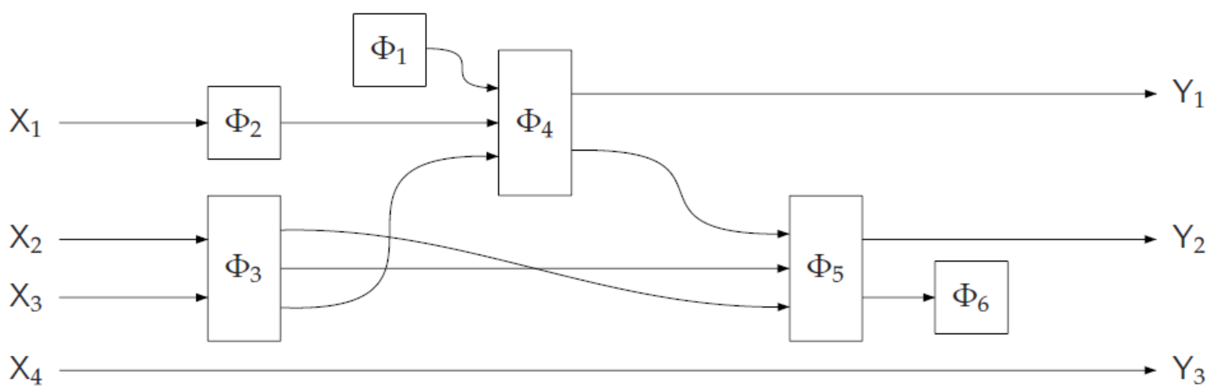

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- does the answer depend on the choices?? **YES!!**
- compare to “circuit depth” for spin chain

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- **complexity** = minimum number of gates required to prepare the desired state

- does the answer depend on the choices?? **YES!!**
- but what does this really mean in quantum field theory? **???**

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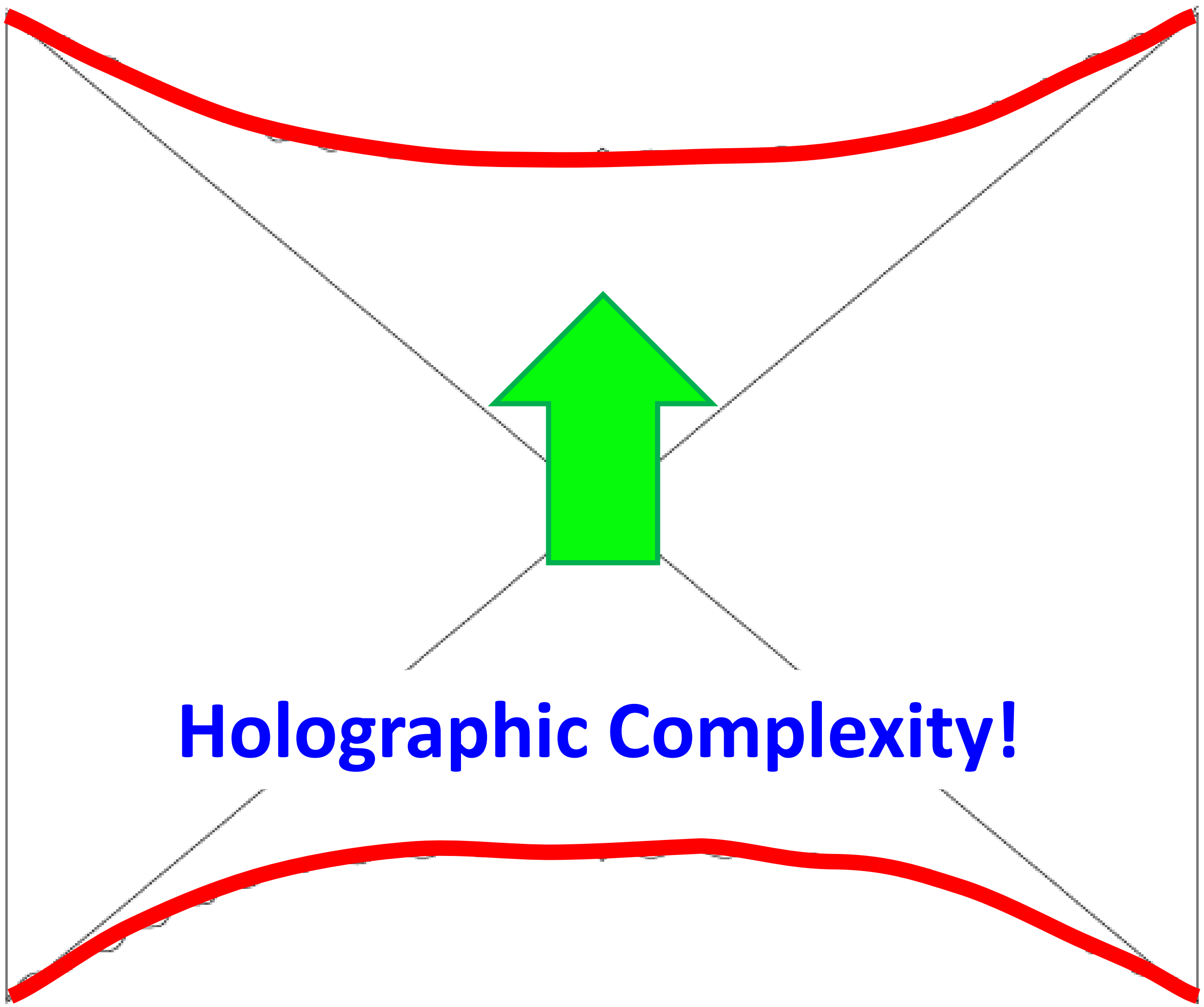
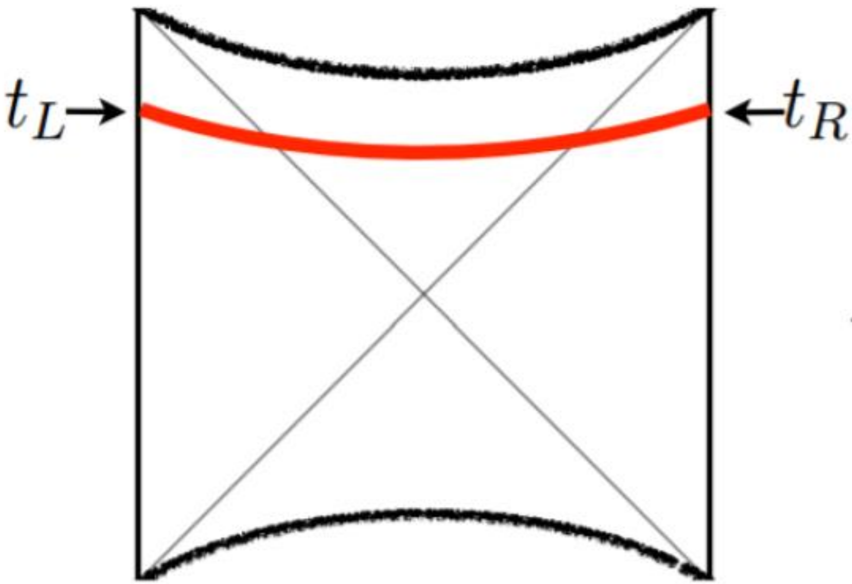


A Tale of Two Dualities: Holographic Complexity

Complexity = Volume



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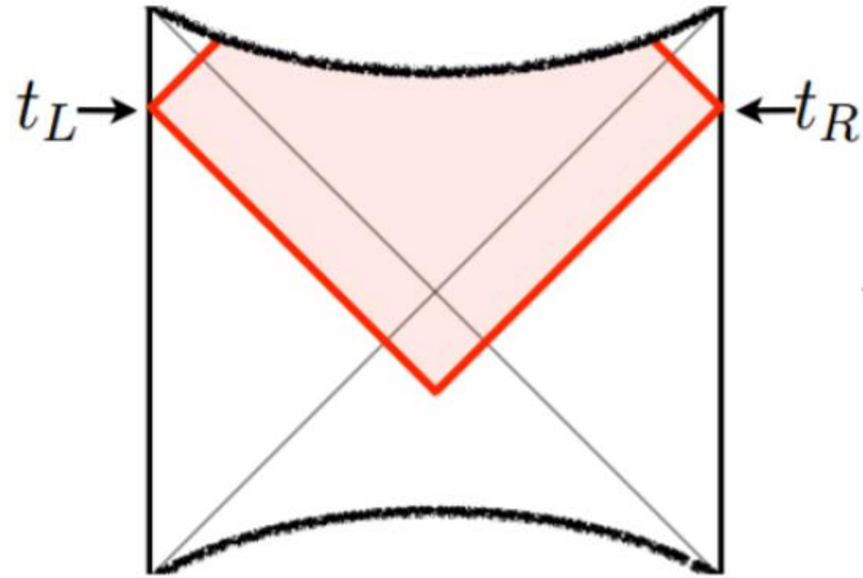

$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N\, \ell} \right]$$

$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\, \hbar}$$

Team Lenny, including Brown, Roberts, Swingle, Stanford, Susskind & Zhao

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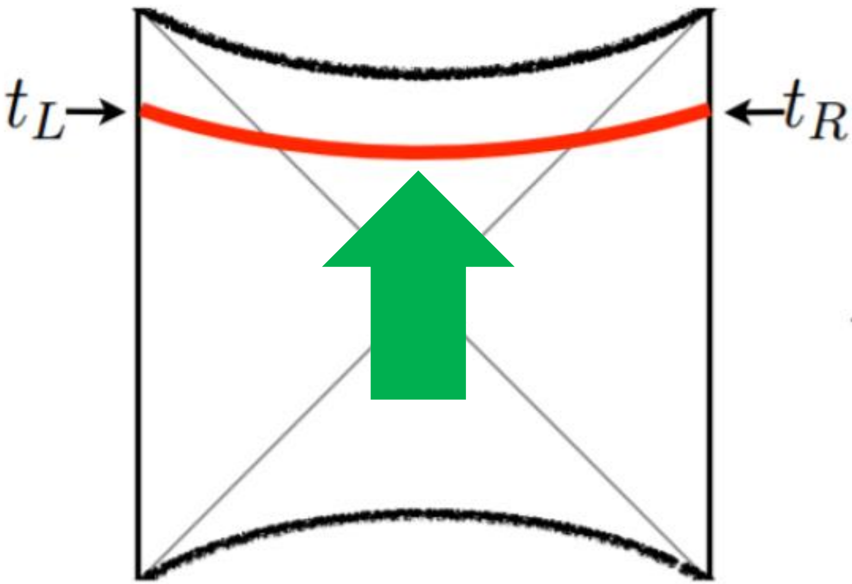

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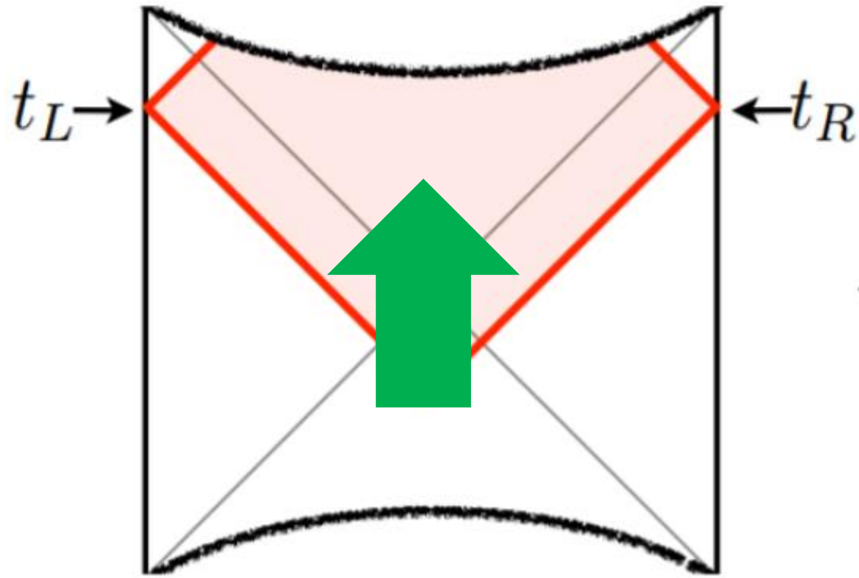

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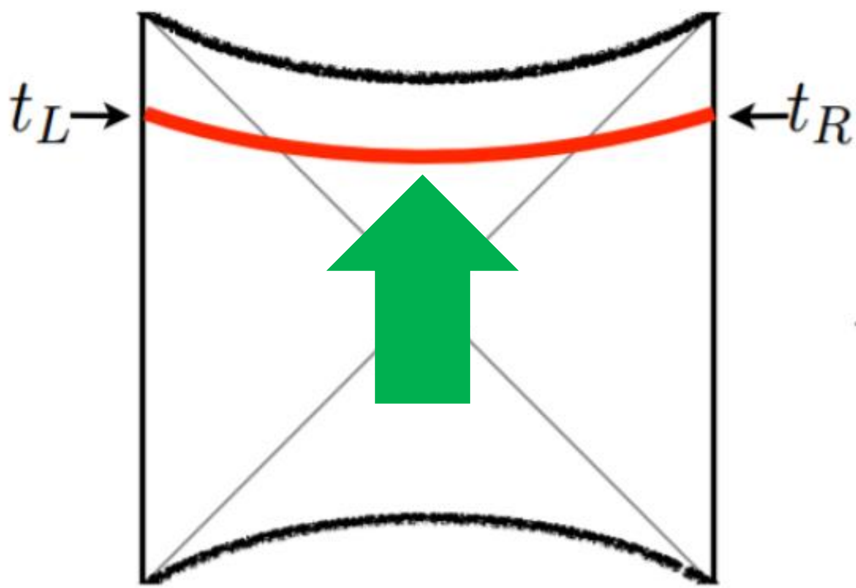

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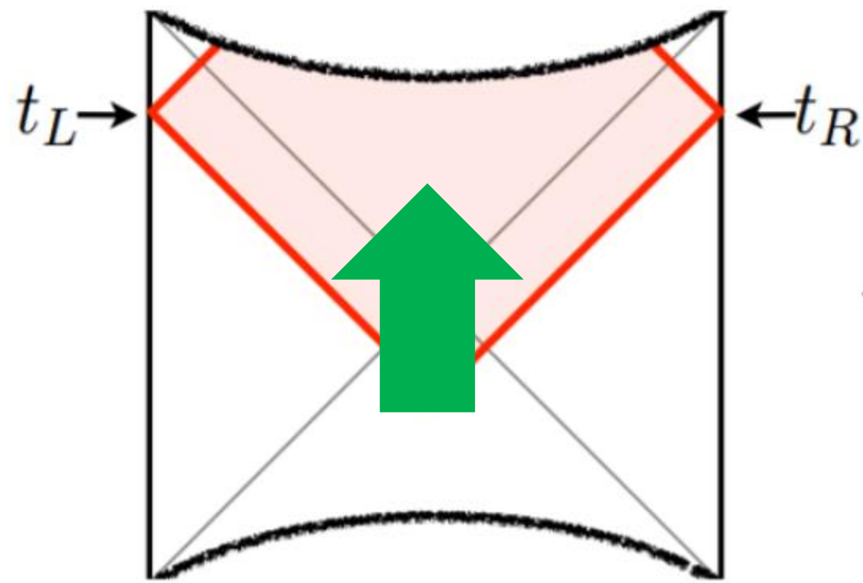

$$\left.\frac{d\mathcal{C}_{\mathrm{V}}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}M$$ (planar)

$(d = \text{boundary dimension})$

$$\left.\frac{d\mathcal{C}_{\mathrm{A}}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

(universal; Lloyd bound)

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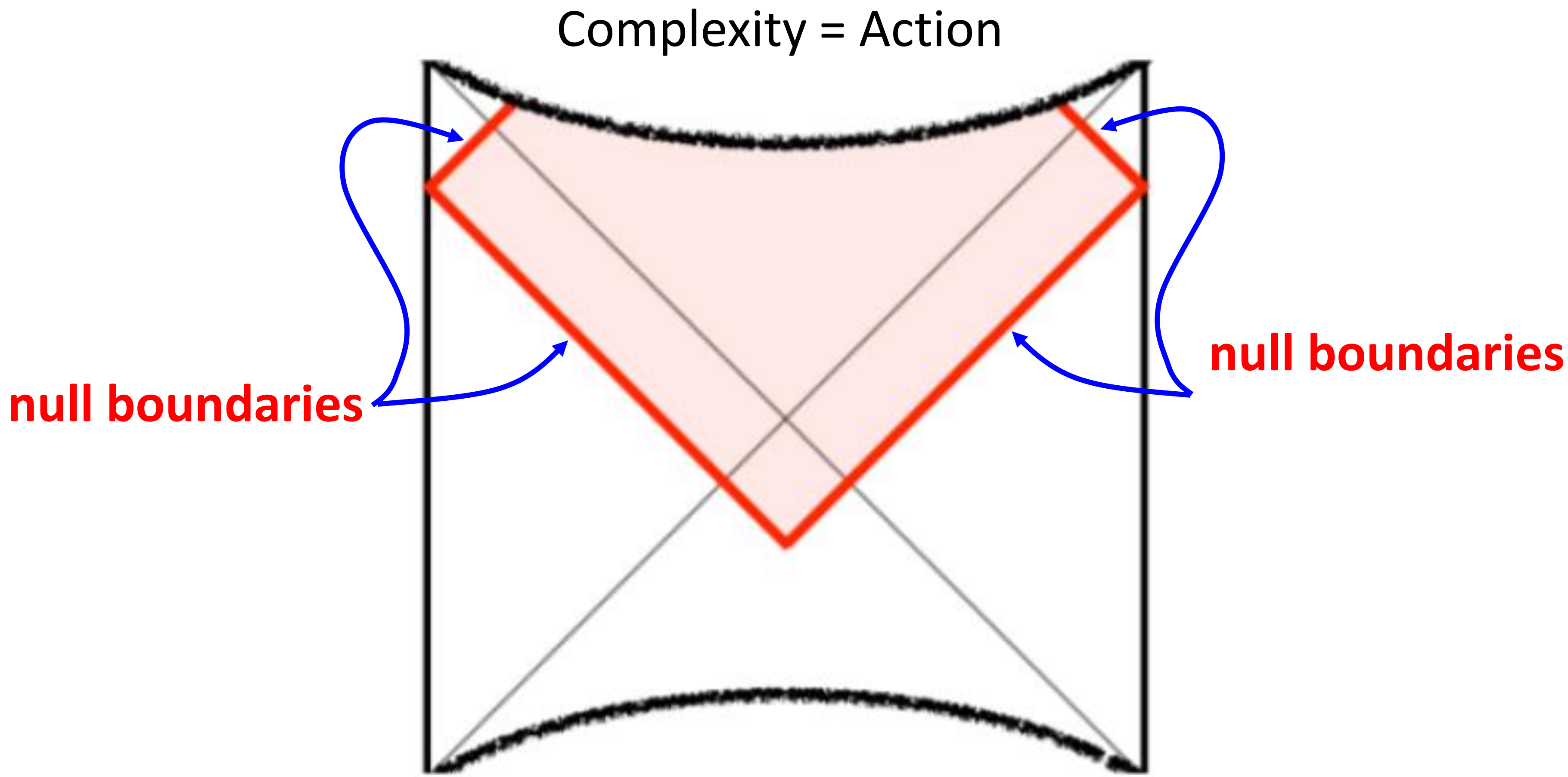

Wheeler-DeWitt patch:

domain of dependence of Cauchy surface ending on boundary time slice

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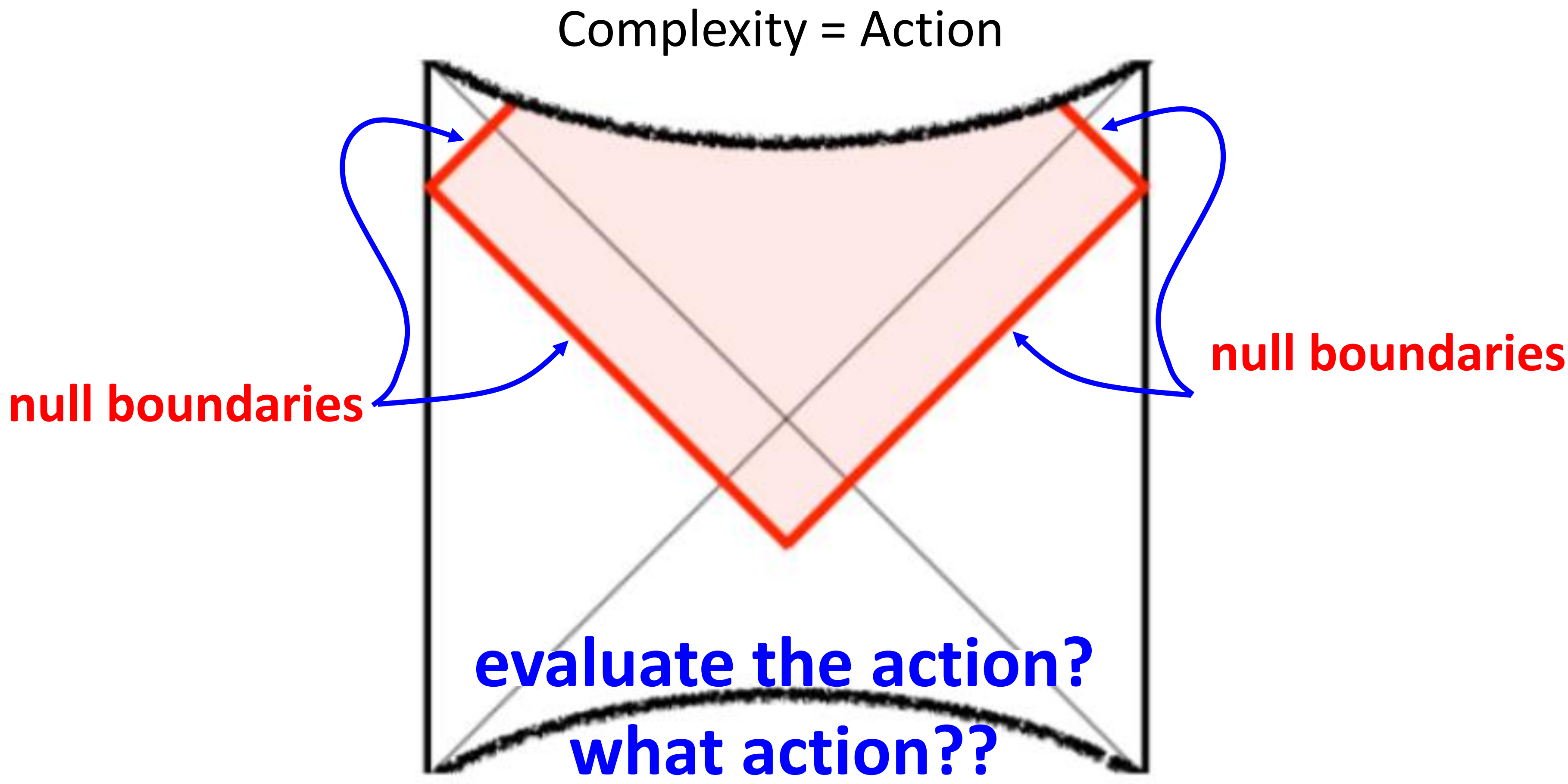

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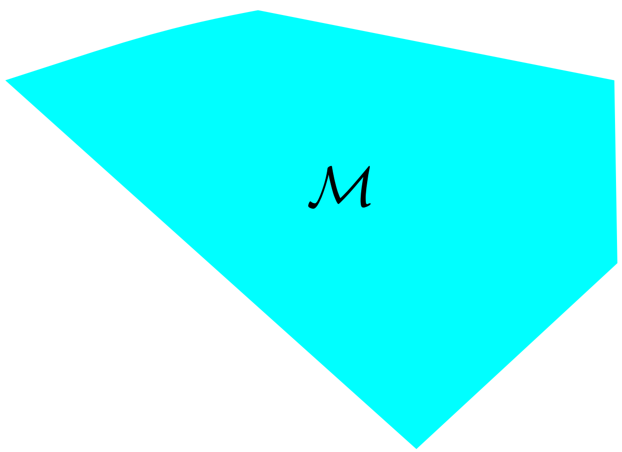

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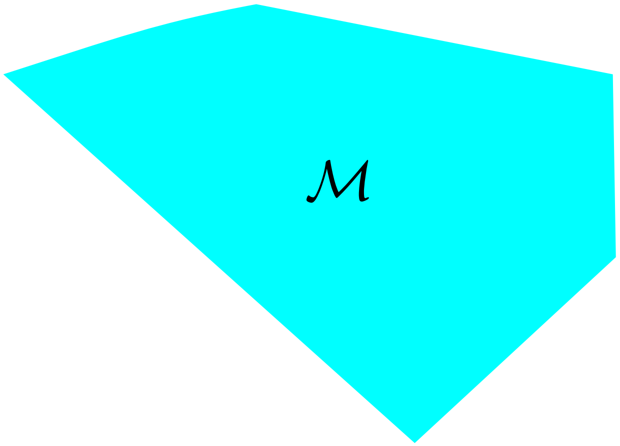

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Gibbons-Hawking-York

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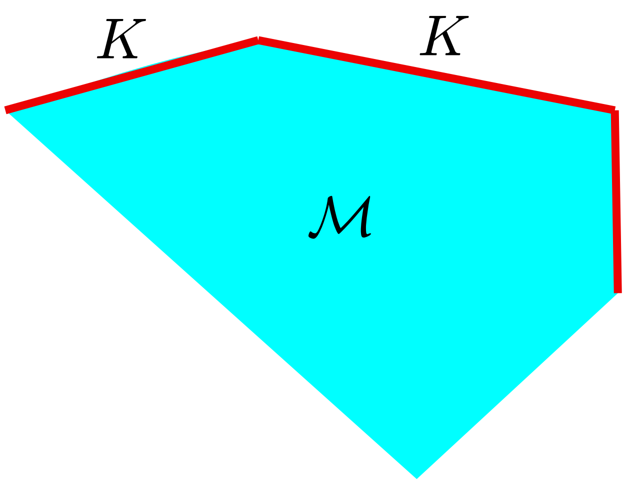

$$K = h^{ab}\nabla_a\mathbf{n}_b$$

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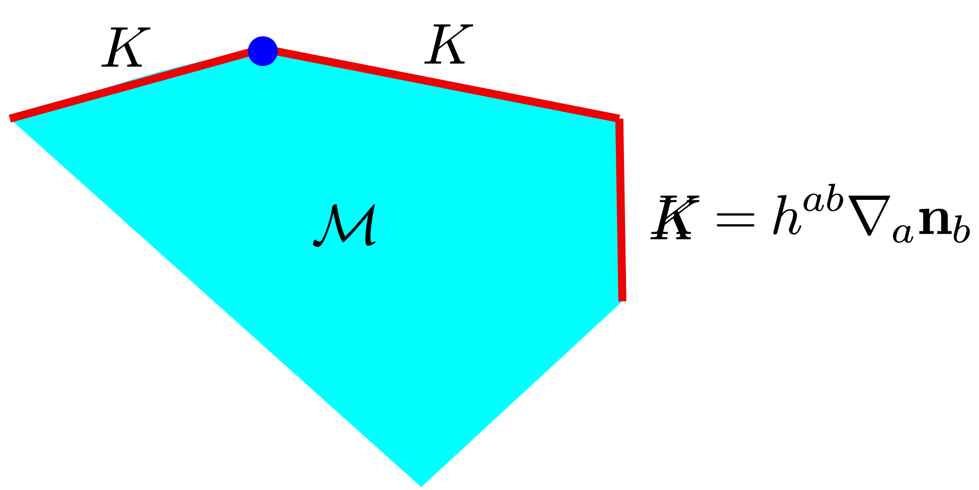
Gibbons-Hawking-York

eg, $$-\int_{\mathcal{M}}(\nabla\Phi)^2 = \int_{\mathcal{M}}\Phi\nabla^2\Phi - \int_{\partial\mathcal{M}}\Phi\,\mathbf{n}\cdot\nabla\Phi$$

- well-defined action principle requires boundary terms



Luis Lehner, RCM, Eric Poisson & Rafael Sorkin

Complexity = Action:

- gravitational action:

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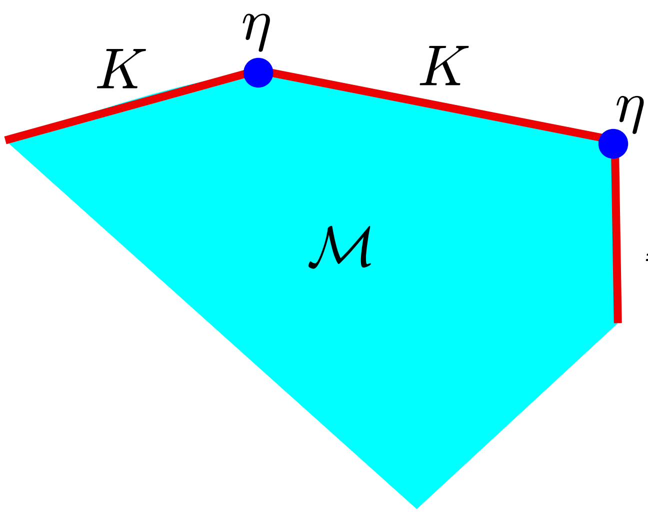
Hayward

- well-defined action principle requires boundary terms



$$K = h^{ab}\nabla_a \mathbf{n}_b$$

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\end{aligned}
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eg,

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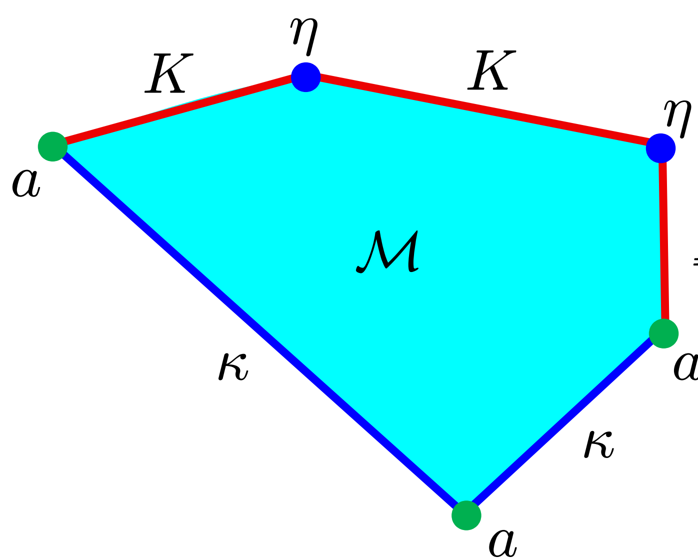

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+ ??????

- ambiguities: total derivatives, extra boundary terms, ...

Luis Lehner, RCM, Eric Poisson & Rafael Sorkin

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⟶ ambiguities in circuit complexity?

Luis Lehner, RCM, Eric Poisson & Rafael Sorkin

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- ambiguities:

1) $k^\nu\, \nabla_\nu\, k^\mu = \boldsymbol{\kappa}\, k^\mu$

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Luis Lehner, RCM, Eric Poisson & Rafael Sorkin

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Aside: make choices without referring to particular metric/coordinates, which allow for meaningful comparison of different states/geometries

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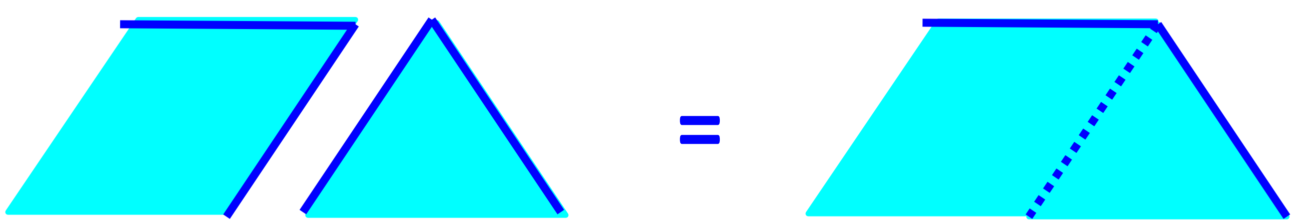

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$$-\ \frac{}{8\pi G_N}\int_{\mathcal{B}'} d\lambda\, d \cdots \sqrt{\gamma}\,\kappa + \frac{}{8\pi G_N}\int_{\Sigma'} d^{d-1}x\sqrt{\sigma}\,a$$

$$\left.\frac{d\mathcal{C}_{\rm A}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

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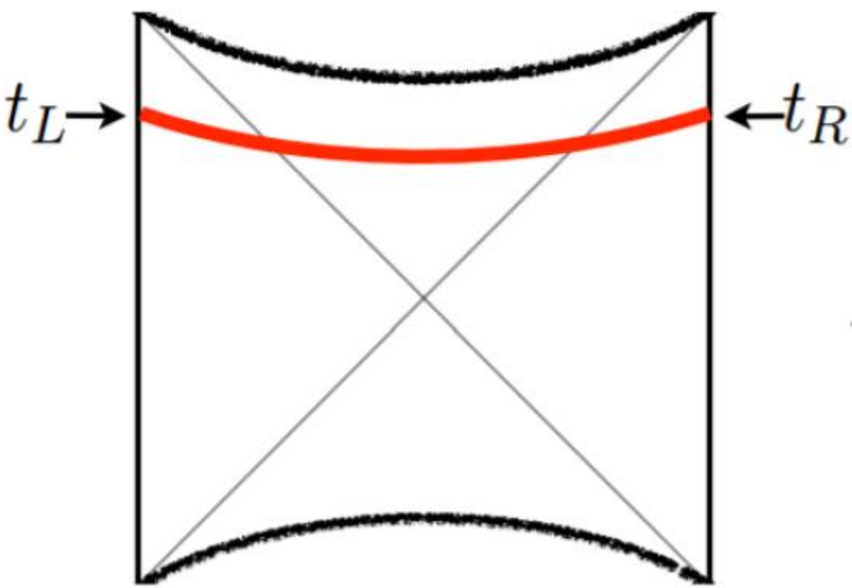
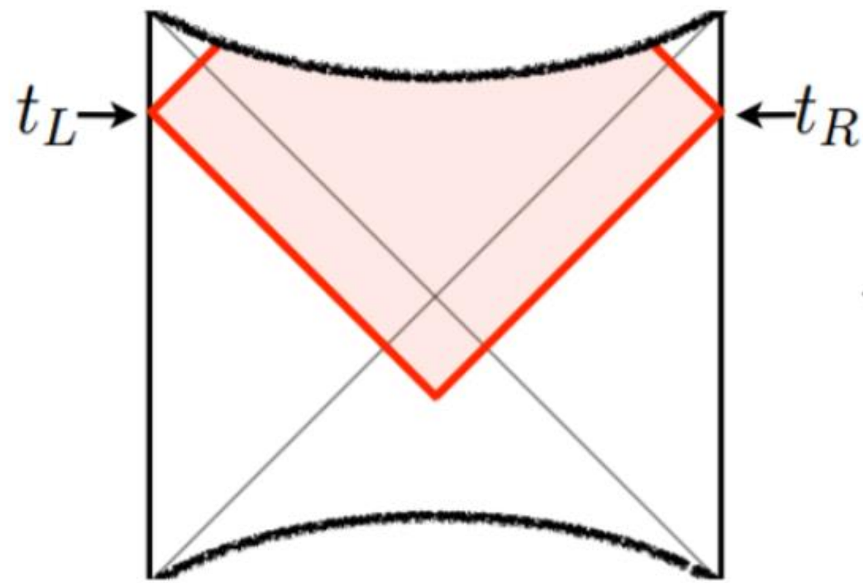
Holographic Complexity:

Complexity = Volume



Complexity = Action



$$\frac{d\mathcal{C}_{\rm V}}{dt}\Big|_{t\to\infty} = \frac{8\pi}{d-1} M$$ (planar)

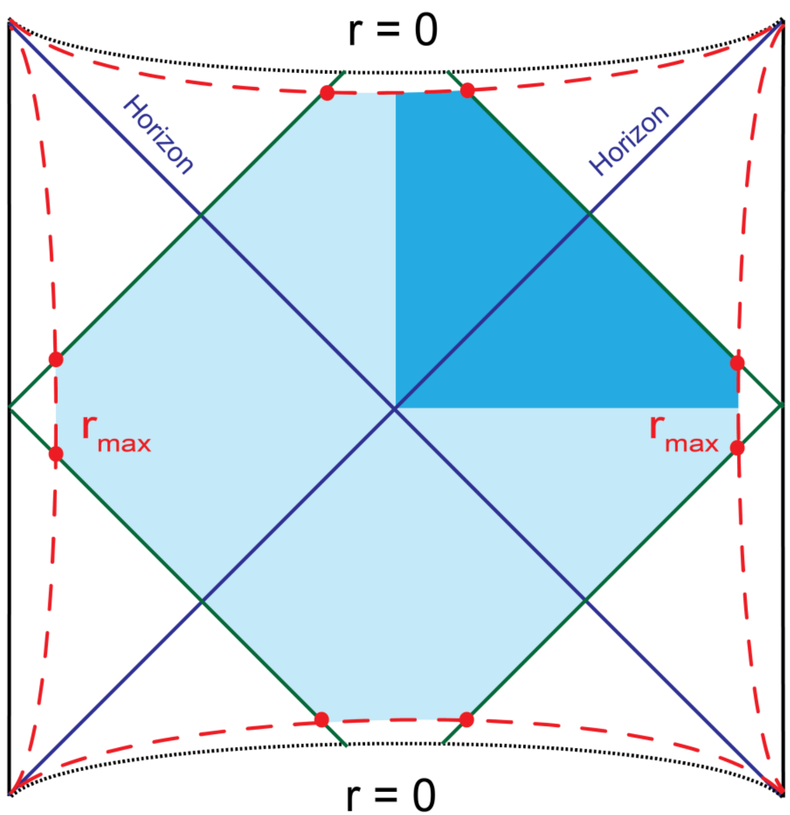
$$\frac{d\mathcal{C}_{\rm A}}{dt}\Big|_{t\to\infty} = \frac{2M}{\pi}$$

Team Lenny, including Brown, Roberts, Swingle, Stanford, Susskind & Zhao

Complexity of Formation:

Shira Chapman, Hugo Marrochio & RCM

$$|\mathrm{TFD}\rangle = Z^{-1/2} \sum_{\alpha} e^{-E_\alpha/(2T)} |E_\alpha\rangle_L |E_\alpha\rangle_R$$



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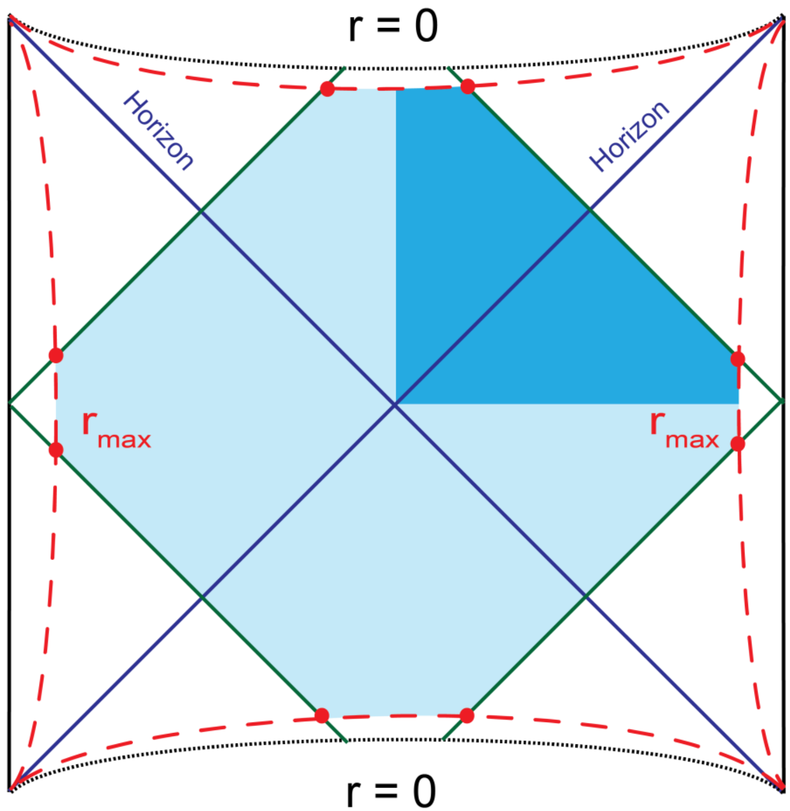
- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state?

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - \mathcal{C}(|0\rangle \quad |0\rangle)$$



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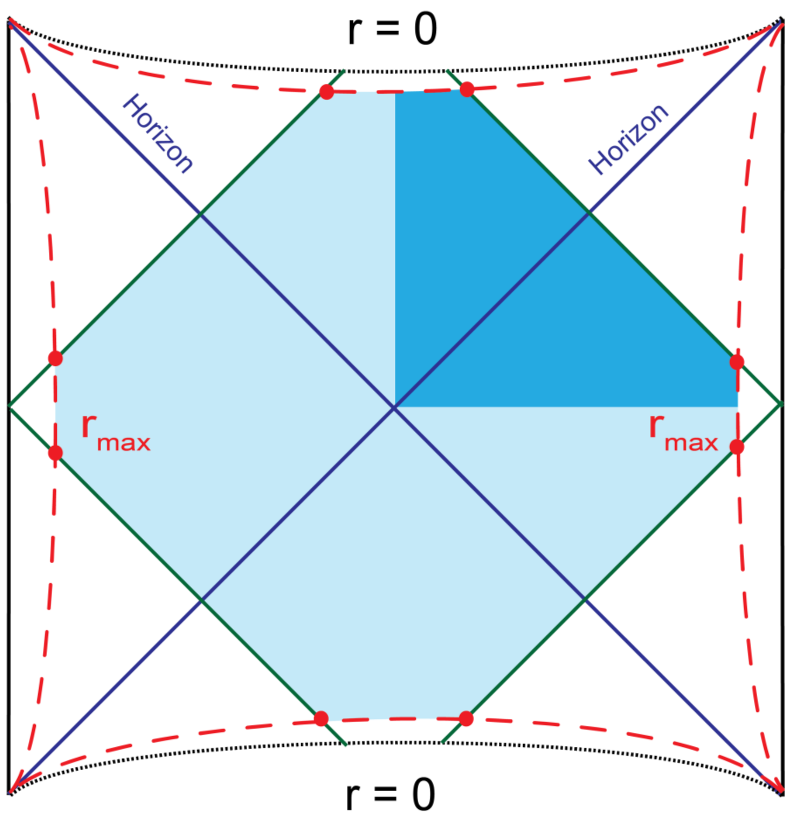
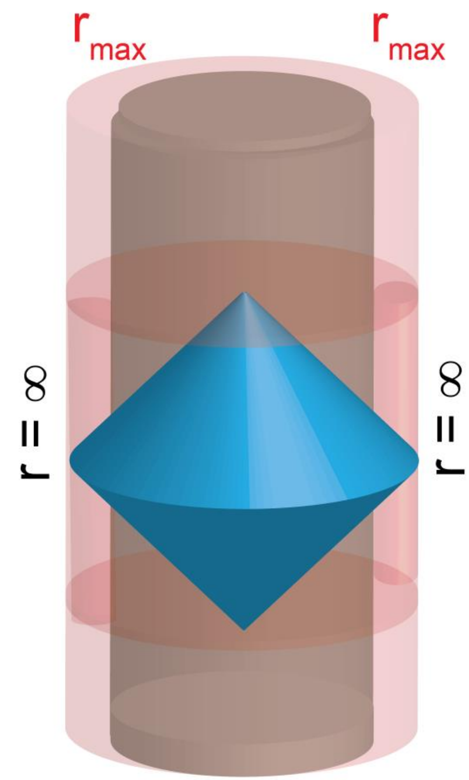
$\Delta\mathcal{C}_A \; =$



$- \; 2\times$



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$$\Delta\mathcal{C}_A = \frac{d-2}{d\,\pi}\,\cot\left(\frac{\pi}{d}\right) S + \cdots$$

curvature corrections

thermal/ent. entropy

$(d = \text{boundary dimension})$

Complexity of Formation:

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S: thermal/ent. entropy

$\cdots$: curvature corrections

$(d = \text{boundary dimension})$

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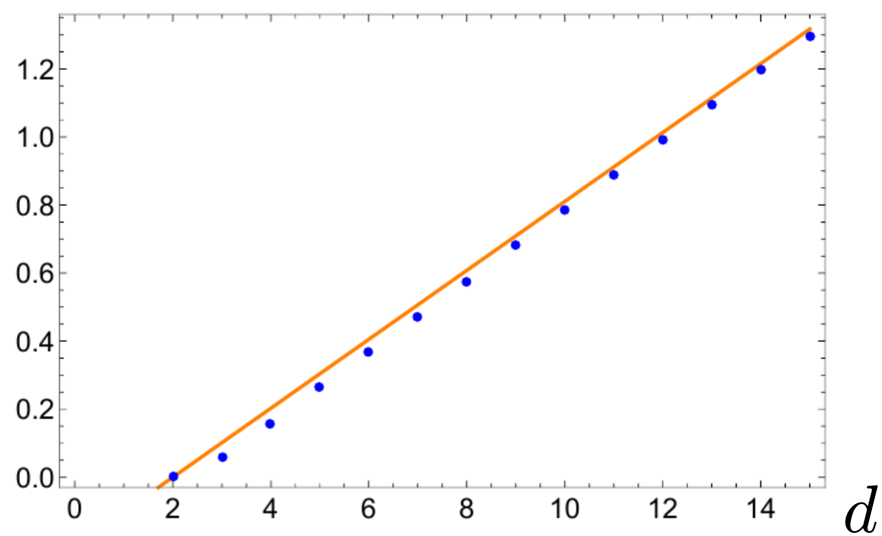
$\frac{d-2}{\pi^2} + \mathcal{O}(1/d)$

thermal/ent. entropy

curvature corrections

• $\frac{d-2}{d\,\pi}\cot\left(\frac{\pi}{d}\right)$

— $\frac{d-2}{\pi^2}$



(d = boundary dimension)

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$\frac{d-2}{\pi^2} + \mathcal{O}(1/d)$

thermal/ent. entropy

curvature corrections

$$\Delta\mathcal{C}_V = 4\sqrt{\pi}\,\frac{(d-2)\,\Gamma(1+\frac{1}{d})}{(d-1)\,\Gamma(\frac{1}{2}+\frac{1}{d})} S + \cdots$$

$4 + \mathcal{O}(1/d)$

$(d = \text{boundary dimension})$

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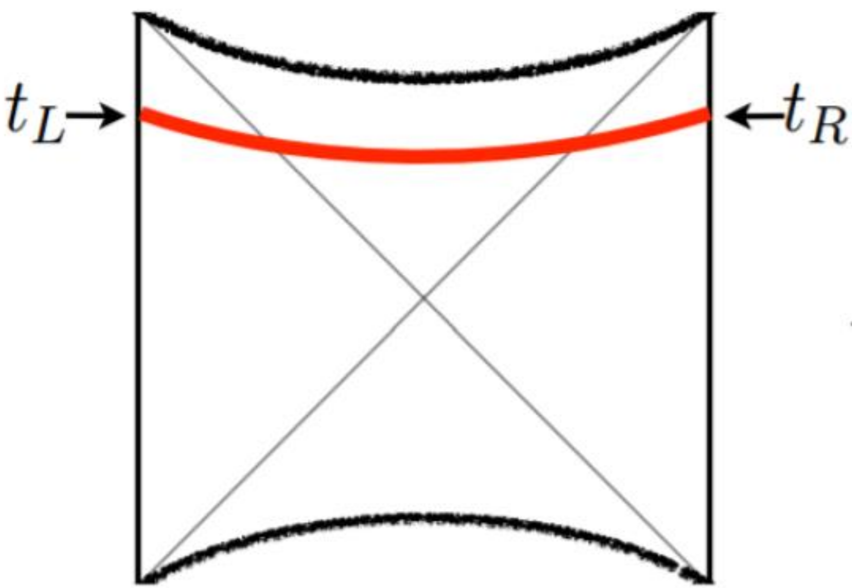
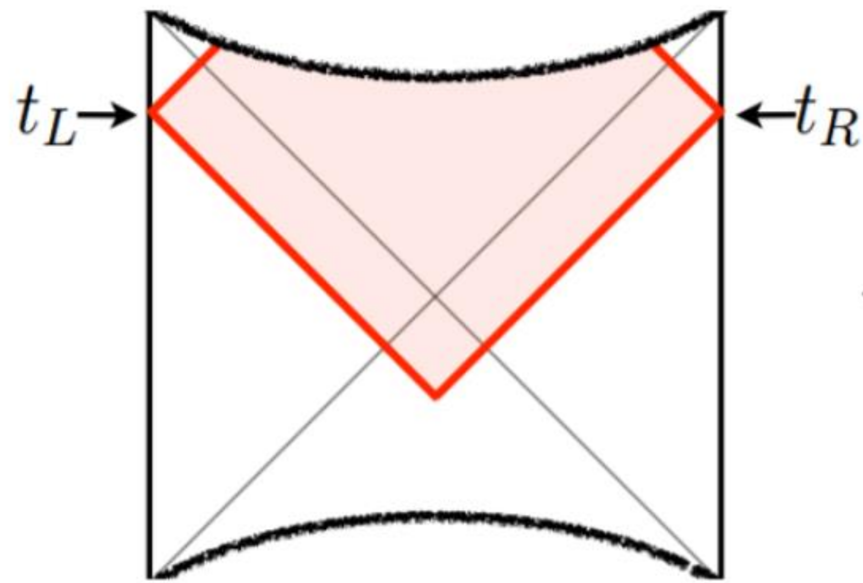
Complexity = Volume



Complexity = Action



$$\left.\frac{d\mathcal{C}_{\mathrm{V}}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}M$$ (planar)

$$\left.\frac{d\mathcal{C}_{\mathrm{A}}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

Team Lenny, including Brown, Roberts, Swingle, Stanford, Susskind & Zhao

Compare?

$$R_{\mathrm{form}} = \frac{\Delta\mathcal{C}_A}{\Delta\mathcal{C}_V} = \frac{d-1}{4\pi^{3/2}}\,\frac{\Gamma\left(1-\frac{1}{d}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{d}\right)} \simeq \frac{d}{4\pi^2}$$

$$R_{\mathrm{rate}} = \frac{d\mathcal{C}_A/dt}{d\mathcal{C}_V/dt} = \frac{d-1}{4\pi^2} \simeq \frac{d}{4\pi^2}$$

$$R_{\mathrm{rate}} - R_{\mathrm{form}} = \frac{\log 2}{2\pi^2} + \mathcal{O}(1/d)$$

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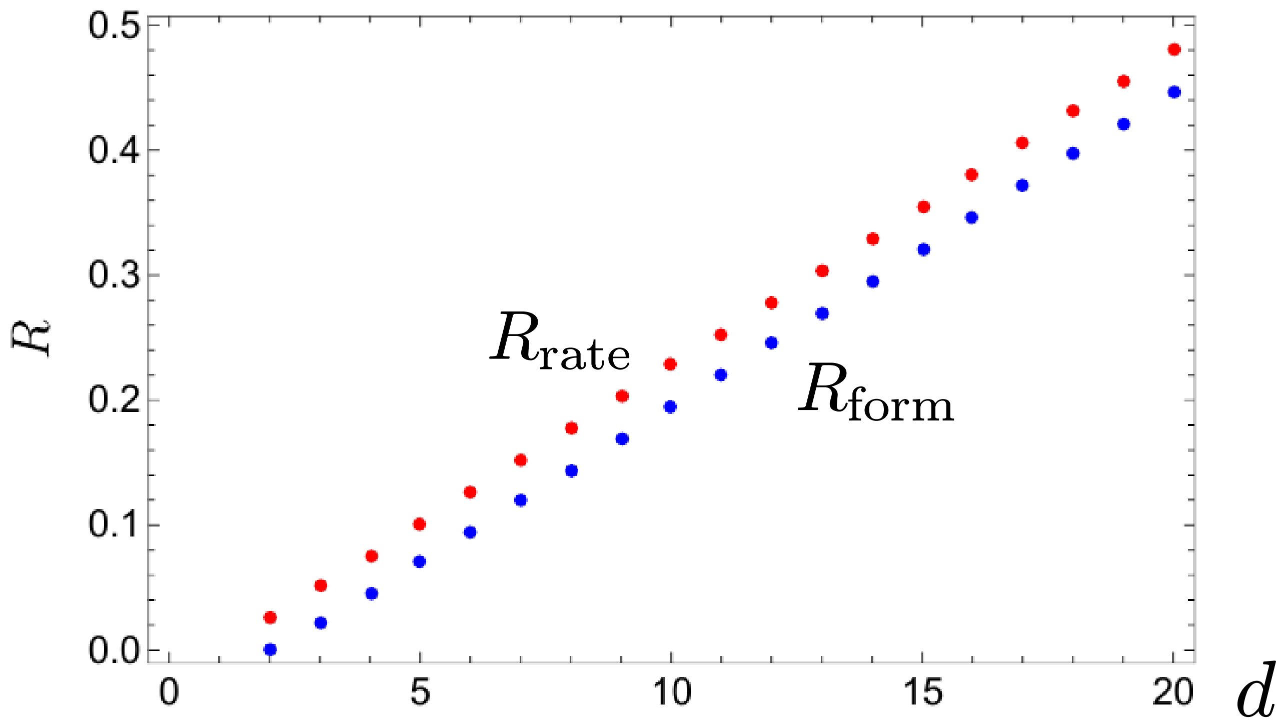

($d =$ boundary dimension)

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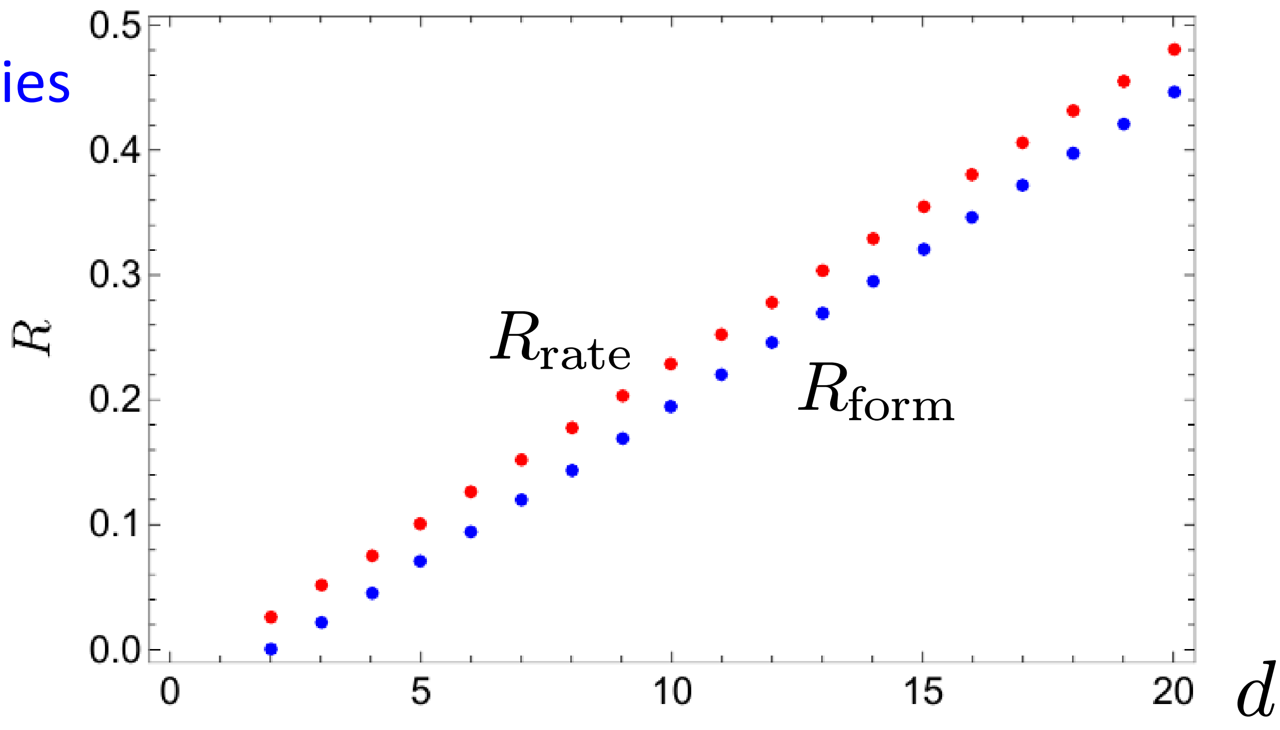
points to consistency of C=V and C=A dualities up to differences in microscopic rules, eg, gate set



(d = boundary dimension)

Complexity of Formation for d=2:

- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state?

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - \mathcal{C}(|0\rangle \quad |0\rangle)$$

$$\Delta\mathcal{C}_A = \frac{d-2}{d\,\pi}\,\cot\left(\frac{\pi}{d}\right) S + \cdots \qquad \underset{d=2}{\simeq} 0!$$

leading term vanishes

$$\Delta\mathcal{C}_V = 4\sqrt{\pi}\,\frac{(d-2)\,\Gamma(1+\frac{1}{d})}{(d-1)\,\Gamma(\frac{1}{2}+\frac{1}{d})}\,S + \cdots \qquad \underset{d=2}{\simeq} 0!$$

- actually holographic calculations apply for $d \geq 3$ (but still correct)

$(d = \text{boundary dimension})$

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- reconsider holographic calculations for 3D BTZ black hole:

$$\Delta\mathcal{C}_A = -\frac{c}{3} \qquad\qquad \Delta\mathcal{C}_V = +\frac{8\pi}{3}\,c$$

$c =$ central charge of boundary CFT

- perhaps related to BTZ black hole being locally AdS geometry

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$$= \ 0 \qquad\qquad\qquad = \ 0 \qquad \text{(R vac)}$$

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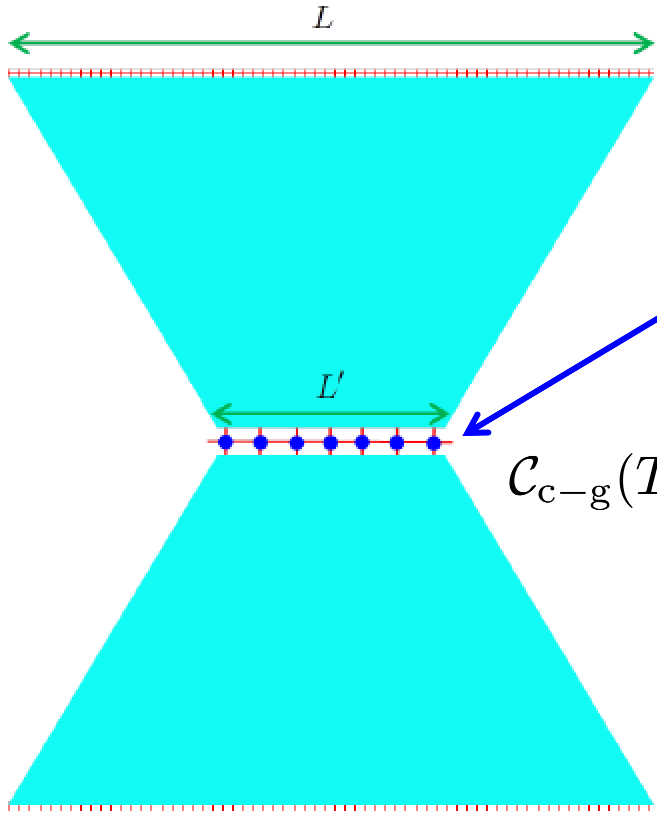
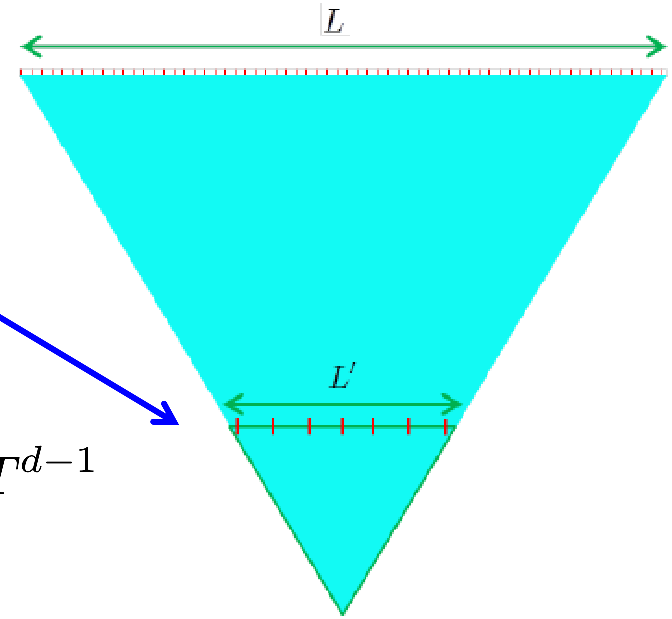
Complexity of Formation from MERA?



$$L' = 2L/\beta$$

$$\mathcal{C}_{\text{c}-\text{g}}(TFD) \simeq C_T \mathcal{V}\, T^{d-1}$$
$$= k_T\, S$$



$$\mathcal{C}_{\text{c}-\text{g}}(vac) \simeq C_T \mathcal{V}\, T^{d-1}$$
$$= k_0\, S$$

$$\Delta\mathcal{C} = \mathcal{C}(TFD) - 2\,\mathcal{C}(vac) = (k_T - 2\,k_0)\, S \;\overset{?}{\underset{?}{\geq}}\; 0 \;{}^{?}_{?}$$

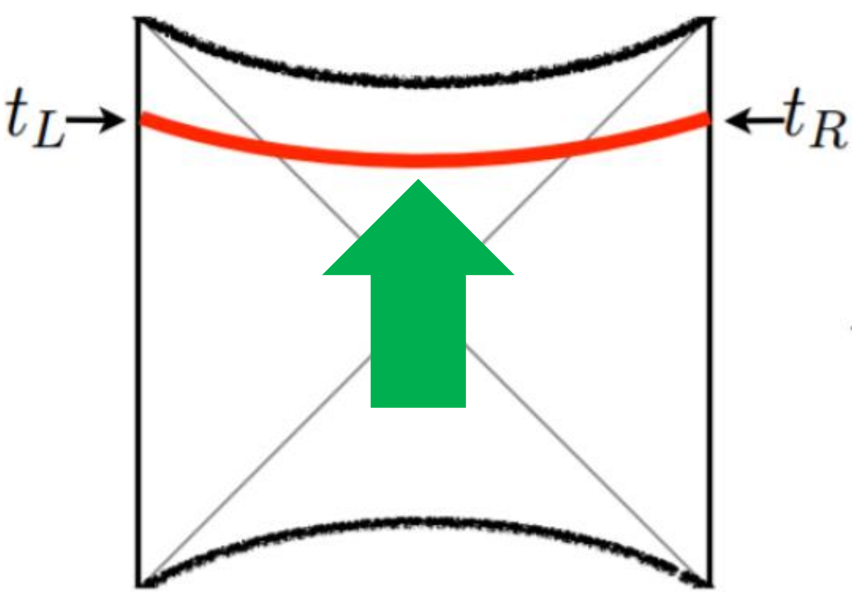
(a) Thermofield double state

(b) Vacuum state

Dean Carmi, RCM & Pratik Rath

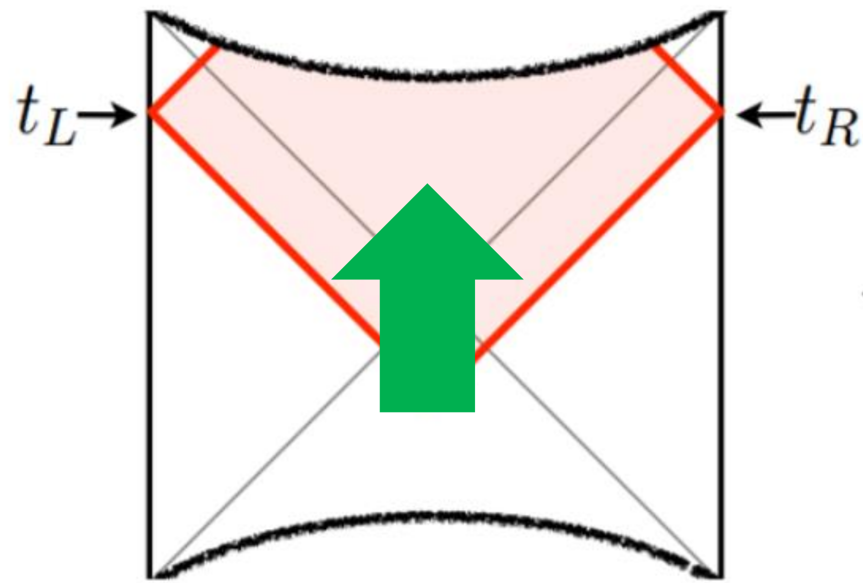
Holographic Complexity:

Complexity = Volume



$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}}\left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

Complexity = Action



$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

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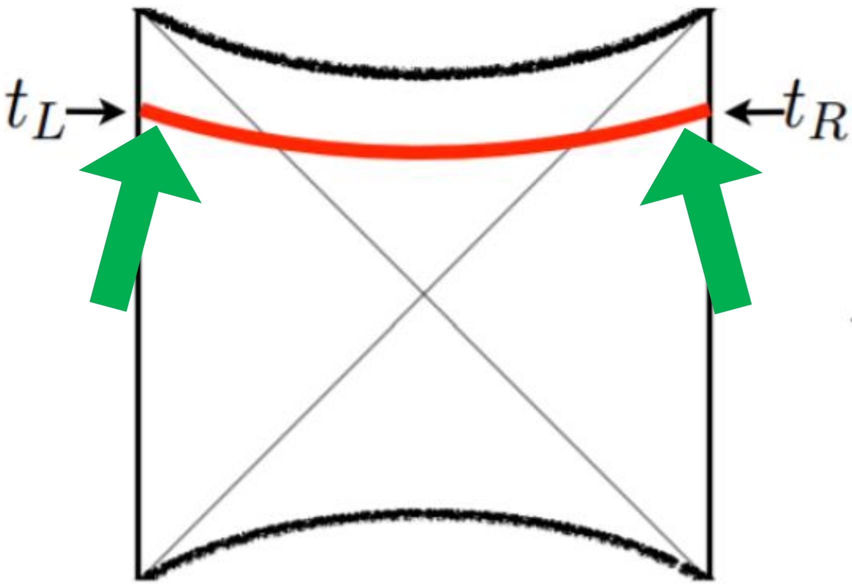

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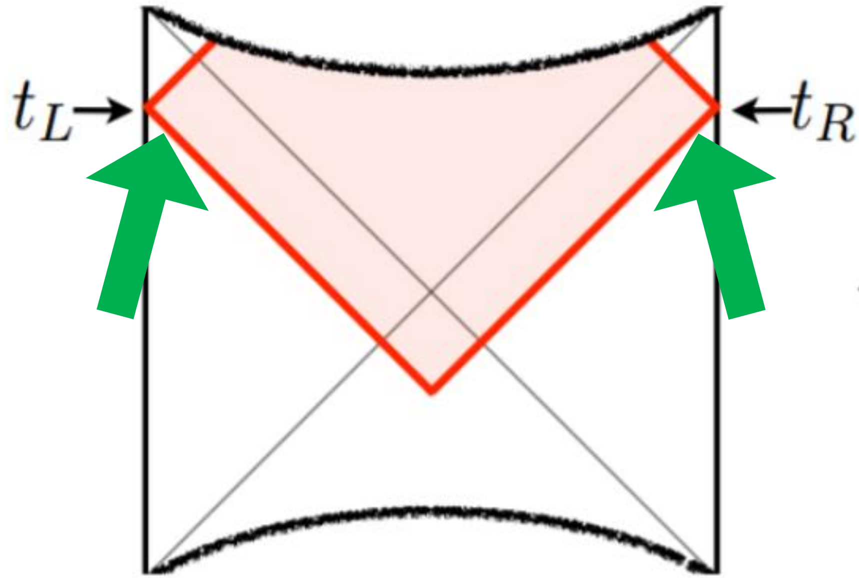

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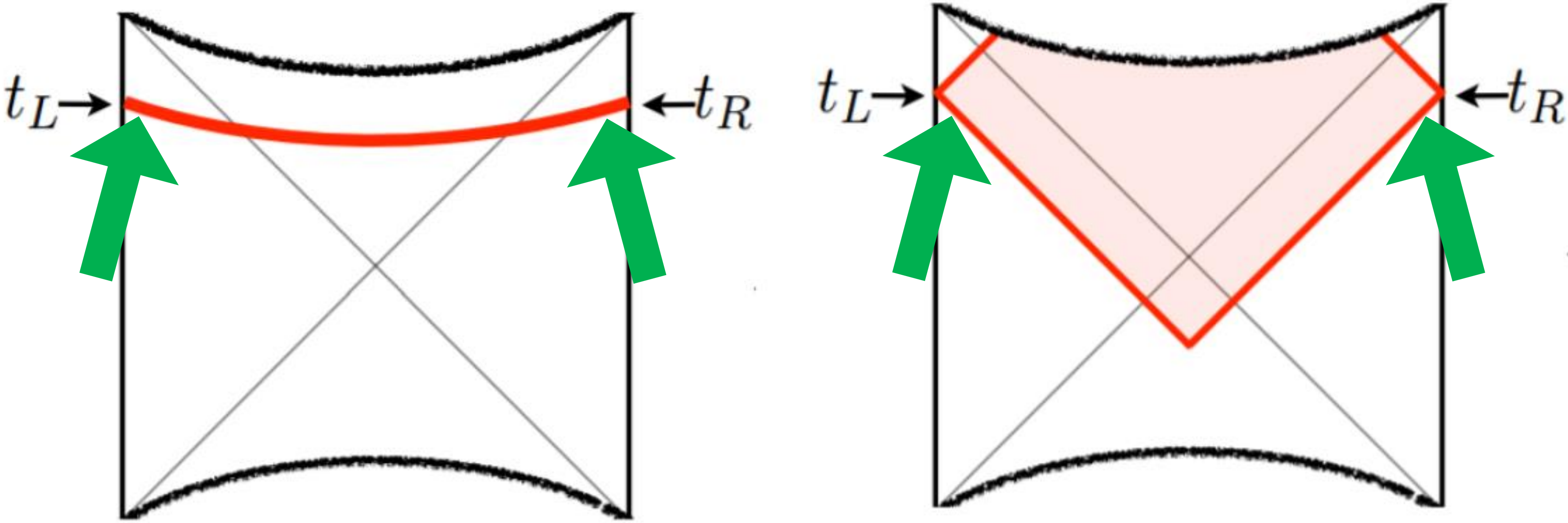
Holographic Complexity:

Dean Carmi, RCM & Pratik Rath

Complexity = Volume　　　　Complexity = Action



- UV divergences naturally associated with establishing correlations or entanglement down to arbitrarily small length scales

Holographic Complexity:

Dean Carmi, RCM & Pratik Rath

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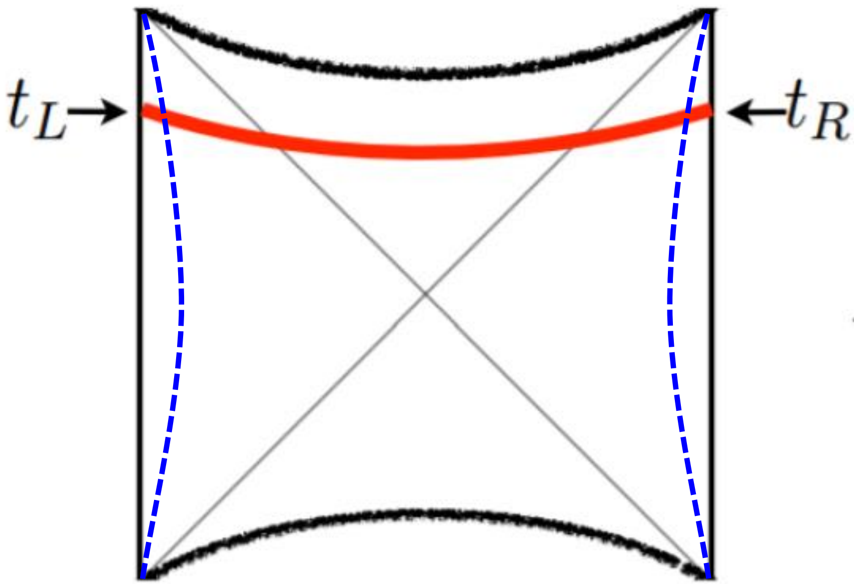

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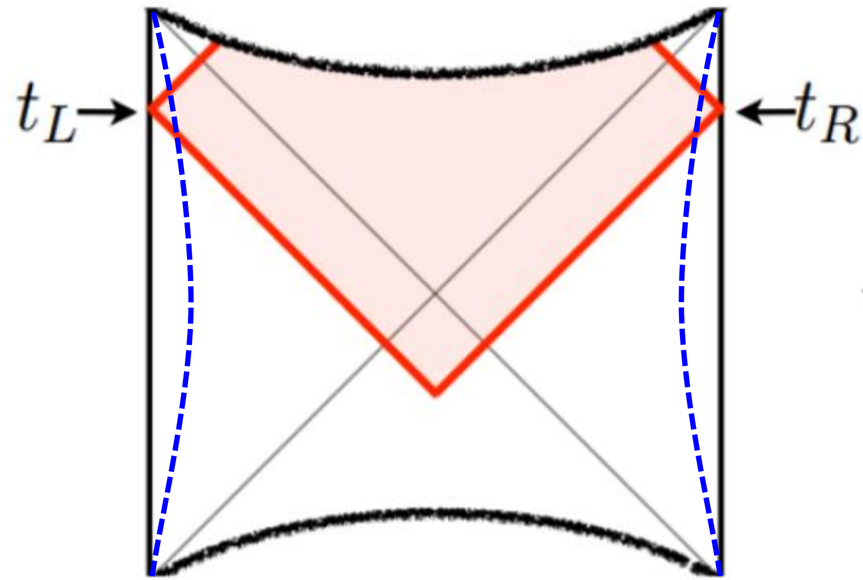

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- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

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$$\mathcal{C}_V(\Sigma) = \frac{L^{d-1}}{(d-1)G_N}\int_\Sigma d^{d-1}\sigma\sqrt{h}\,\Big[\frac{1}{\delta^{d-1}} - \frac{(d-1)}{2(d-2)(d-3)\delta^{d-3}}\left(\mathcal{R}^a_a - \frac{1}{2}\mathcal{R} - \frac{(d-2)^2}{(d-1)^2}K^2\right) + \cdots\Big]$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

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$$\mathcal{C}_V(\Sigma) = \frac{8\pi^{\frac{d+2}{2}}\Gamma(d/2)}{\Gamma(d+2)}\,C_T \int_\Sigma d^{d-1}\sigma\sqrt{h}\left[\frac{1}{\delta^{d-1}} - \frac{(d-1)}{2(d-2)(d-3)\delta^{d-3}}\left(\mathcal{R}_a^a - \frac{1}{2}\mathcal{R} - \frac{(d-2)^2}{(d-1)^2}K^2\right) + \cdots\right]$$

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$$\mathcal{C}_V(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\; v_0(\mathcal{R}, K)$$

$$\mathcal{C}_A(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\left[v_1(\mathcal{R}, K) + \log\left(\frac{L}{\alpha\,\delta}\right)\; v_2(\mathcal{R}, K)\right]$$

with $$v_k(\mathcal{R}, K) = \sum_{n=0}^{\lfloor\frac{d-1}{2}\rfloor} \sum_i c_{i,n}^{[k]}(d)\; \delta^{2n}\; [\mathcal{R}, K]_i^{2n}$$

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AdS scale

normalization

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from asymptotic joint

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Holographic Complexity:

- UV divergences appear as local integrals of geometric invariants

$$\mathcal{C}(\Sigma) \simeq c_0\, \mathcal{V}(\Sigma)/\delta^{d-1} + \cdots$$

$t = \text{constant}$ Σ_1

$|\psi_0\rangle$

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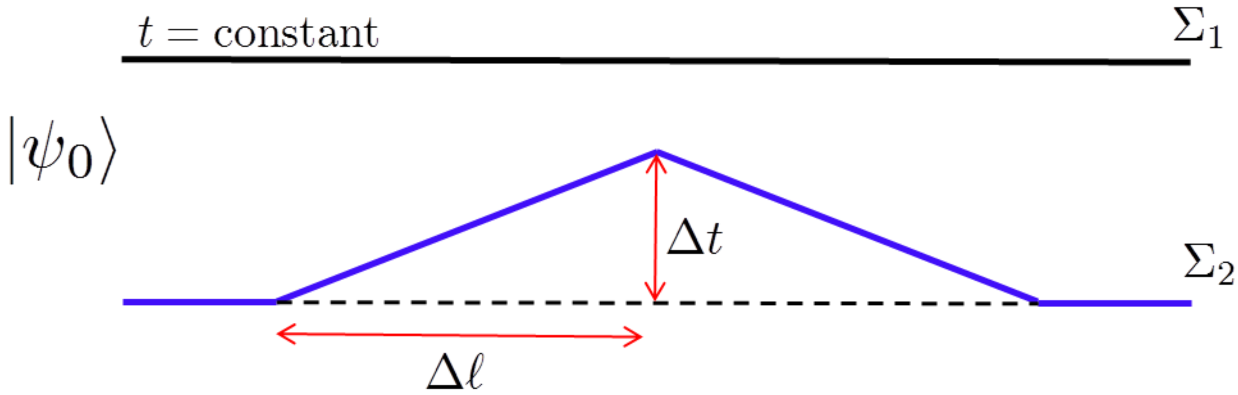

$$\Delta\mathcal{C} \simeq 2c_0\, (\sqrt{\Delta\ell^2 - \Delta t^2} - \Delta\ell)\, \mathcal{V}_{\rm trans}/\delta^{d-1} + \cdots < 0$$

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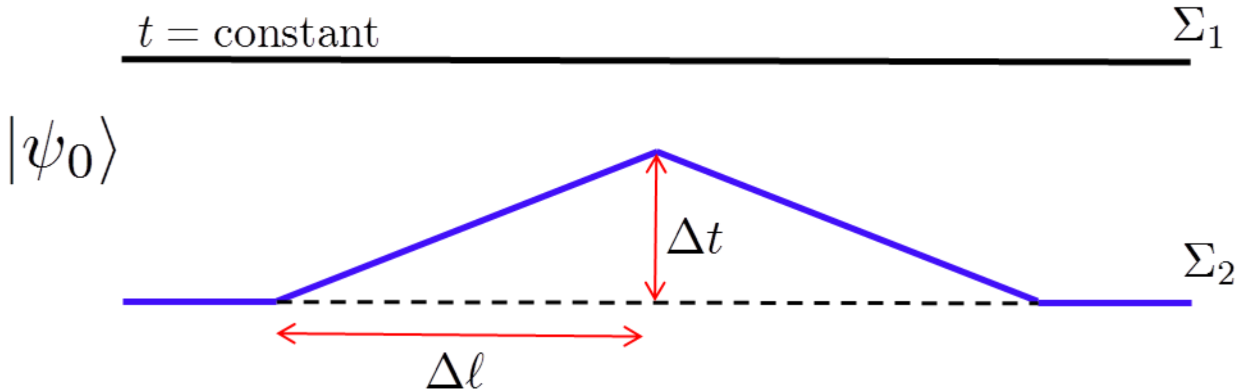

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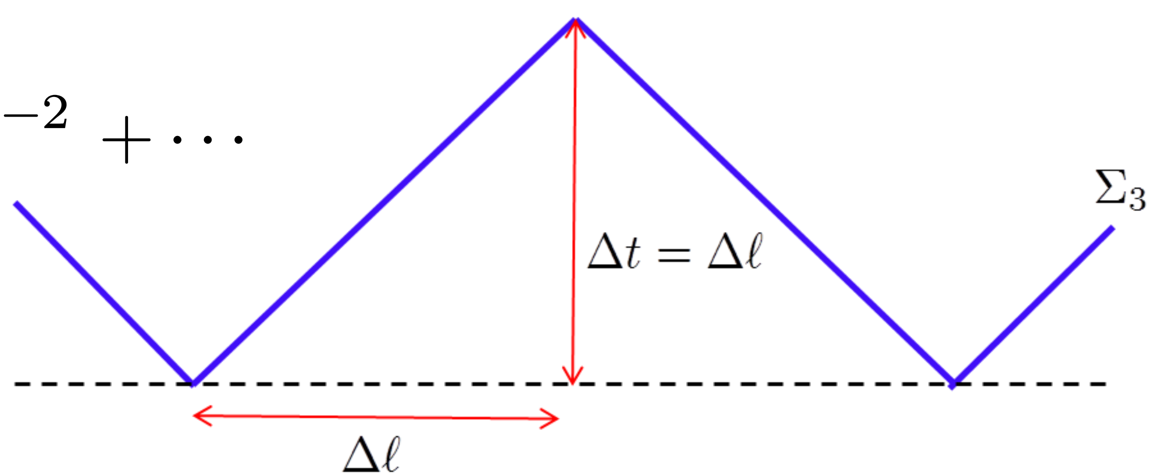
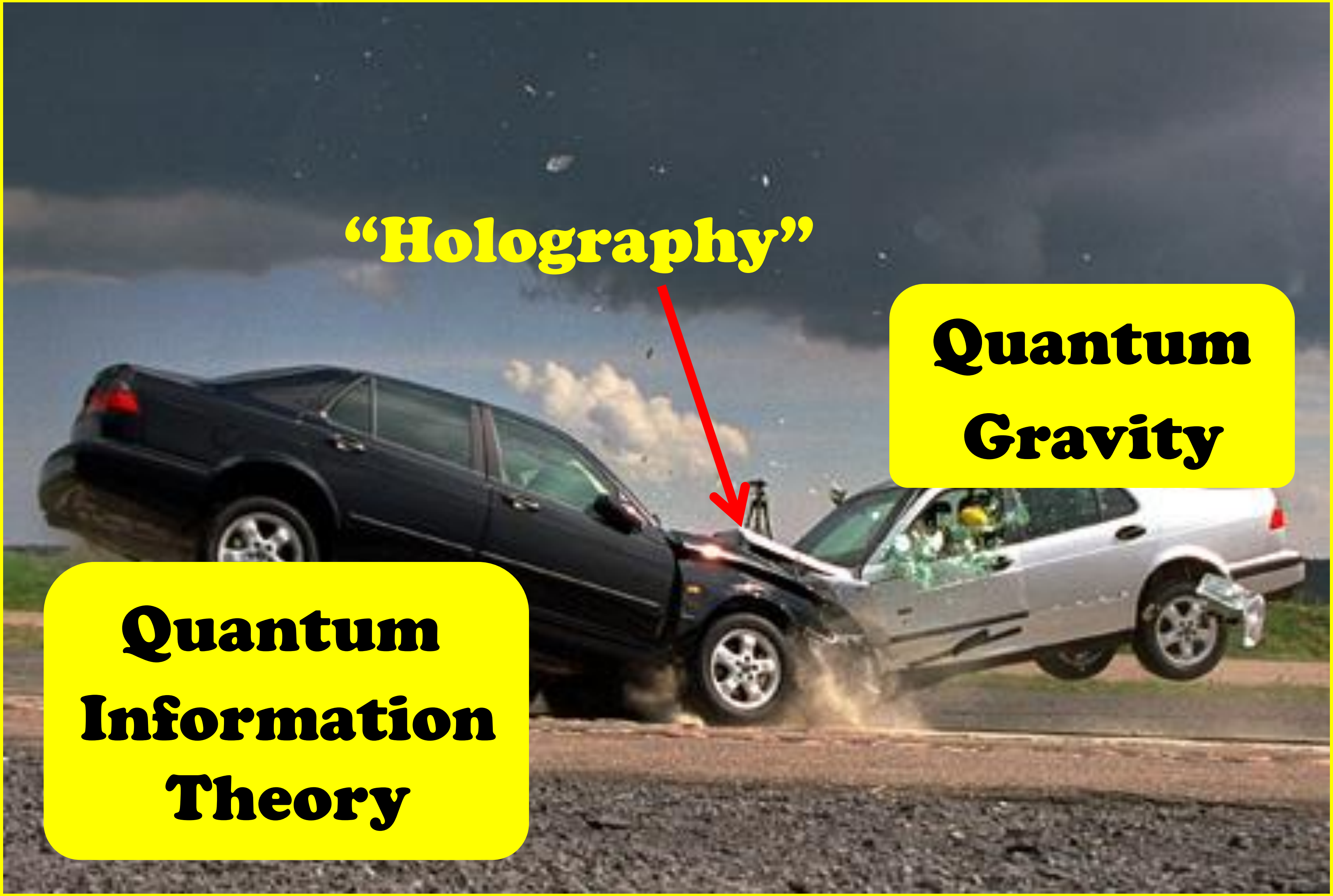
$$\mathcal{C}(\Sigma_3) \simeq c_1\, \mathcal{V}_{\text{trans}}/\delta^{d-2} + \cdots$$



Questions?

- What is "holographic complexity"?
  - QFT/path integral description of "complexity" in boundary CFT?
  - what is boundary dual of these gravitational observables?
- is there a privileged role for (states on) null Cauchy surfaces?
  - provide distinguished reference states?
- is there a "renormalized holographic complexity"?
  - what's it good for?; (EE vs mutual information versions of F)
- ambiguities? ambiguities? ambiguities?
  - connections between ambiguities in gravity and boundary?
- more boundary terms: higher codim. intersections; "complex" joint contributions; boundary "counterterms"
- why is complexity of formation positive?
- $\mathcal{C}_A$ contribution of spacetime singularity?
- subregion complexity?

"It from Qubit": New Collision of Ideas



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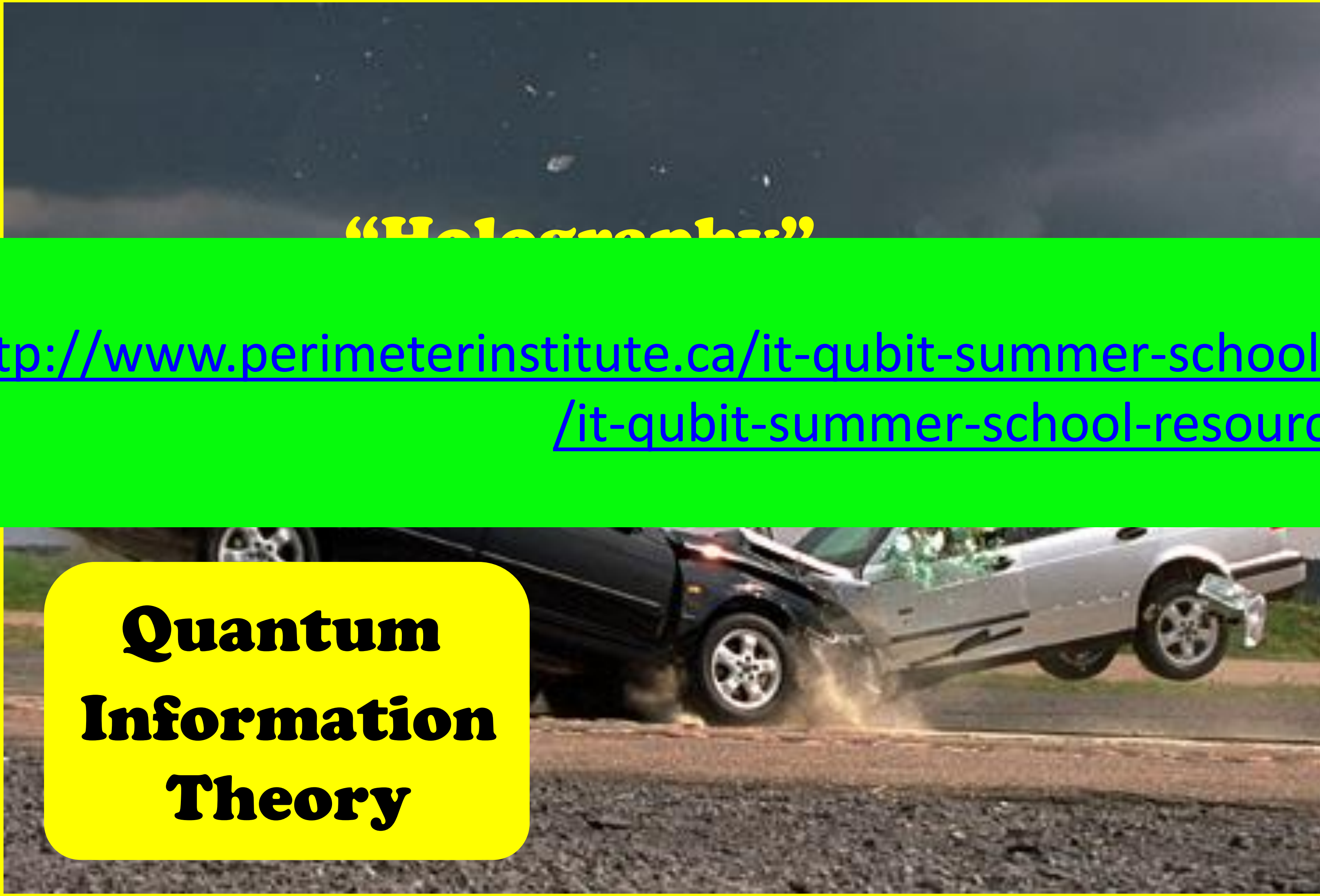

http://www.perimeterinstitute.ca/it-qubit-summer-school/it-qubit-summer-school-resources