Problems for **Entanglement in Quantum Field Theory and Holography**
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XIX Swieca School
Maresias, Brazil; January 31 to February 10, 2017

Probem 1: This first problem is a series of exercises that will help you to become more familiar with density matrices:

Evaluate the reduced density matrices, their eigenvalues and the entanglement entropy resulting when the second photon is traced out of the two photon states introduced in the first lecture:

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{2}}\Big(|+-\rangle - |-+\rangle\Big) \\
|\psi_2\rangle &= \frac{1}{2}\Big(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle\Big) \\
|\psi_2\rangle &= \frac{1}{2}\Big(|++\rangle + |+-\rangle - |-+\rangle + |--\rangle\Big)
\end{aligned}
$$

More generally, consider a global state of the form

$$|\psi\rangle = \sum_{i,j} \alpha_{ij}|ij\rangle \qquad \text{where} \;\; |ij\rangle = |i\rangle \otimes |j\rangle$$

and $|i\rangle$ and $|j\rangle$ are orthonormal states in two separate Hilbert spaces (possibly of different dimensions). The coefficients α_{ij} are complex numbers.

- Evaluate the reduced density matrix ρ that results when the second set of degrees of freedom (the $|j\rangle$'s) are integrated out.
- Show that $\rho^\dagger = \rho$. Recall that this implies that the eigenvalues of ρ are real.
- Show that $\mathrm{Tr}[\rho] = 1$ if the initial state was properly normalized, *i.e.,* $\langle\psi|\psi\rangle = 1$. Recall that this implies the the eigenvalues sum to one, *i.e.,* $\sum \lambda_a = 1$.
- With a bit more work, we can also show that the eigenvalues are also positive or zero. Assuming the latter, show that $\mathrm{Tr}[\rho^2] \le 1$. Further show that the inequality is only saturated when one of the eigenvalues is one, *e.g.,* $\lambda_1 = 1$, and the rest are zero. It follows that $\mathrm{Tr}[\rho^2] = 1$ if and only if ρ describes a pure state.
- For any operator that acts only in the first Hilbert space, *i.e.,* $\mathcal{O} \equiv \mathcal{O} \otimes \mathbb{1}$, show that

$$\langle\psi|\mathcal{O}|\psi\rangle = \mathrm{Tr}[\rho\,\mathcal{O}]\,.$$

Probem 2: Carry out the calculation of the entanglement entropy in the ground state of two coupled simple harmonic oscillators, which I very briefly discussed in the first lecture. Recall that Hamiltonian was given by

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2(x_1^2 + x_2^2) + \Omega^2(x_1 - x_2)^2\right]\,.$$

Hence, you should find that the ground state wavefunction can be written as

$$\Psi(x_1, x_2) = \frac{(\omega_+ \omega_-)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{1}{4}\omega_+(x_1 + x_2)^2 - \frac{1}{4}\omega_-(x_1 - x_2)^2\right]$$

where $\omega_+ = \omega$ and $\omega_- = (\omega^2 + 2\Omega^2)^{1/2}$.

Note that the full calculation can be found in Srednicki's paper [1].

Consider repeating the calculation of the entanglement entropy but in the case where one of the normal modes is in the first excited state.

For students needing a refresher on the solution of the simple harmonic oscillator in quantum mechanics, they might consult: http://en.wikipedia.org/wiki/Quantum_harmonic_oscillator

Problem 3: In the first lecture, I mentioned that the previous calculation is readily extended to a system of many coupled oscillators, e.g., a lattice description of a free scalar field theory. This calculation was first discussed by Bombelli et al [2] and then again by Srednicki [1]. However, I will refer you to a nice review of the calculation is section 2.2 of [3]. There the Hamiltonian is written as

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} \phi_i \, K_{ij} \, \phi_j \ .$$

(Actually they replace p_i with $\dot{\phi}_i$, as is standard in Lagrangian mechanics. Note that the mass of the particles is 1.) The ground state wavefunction is then written as

$$\Psi(\phi_i) = \left(\det \frac{W}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}\phi^T \, W \, \phi\right]$$

where $W = \sqrt{K}$.

Show that this formalism can be applied to describe the previous calculation for two coupled oscillators.

Show that your previous result for the entanglement entropy matches the general expression in eq. (42) of [3].

Problem 4: Consider the scalar field action

$$I = -\frac{1}{2}\int d^dx \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi \, \partial_\nu\phi + m^2\phi^2 + \xi \, R(g) \, \phi^2\right]$$

where $R(g)$ is the Ricci scalar of the background metric $g_{\mu\nu}$. Show that the action is invariant (up to total derivatives, i.e., boundary terms) under local Weyl rescalings

$$g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu} \,, \qquad \phi \to e^{-\Delta\omega(x)}\phi \quad \text{with} \;\; \Delta = \frac{d-2}{2}$$

if $m = 0$ and $\xi = \frac{d-2}{4(d-1)}$.

The stress tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta I}{\delta g^{\mu\nu}} \ .$$

Show that in the case where the mass and the coupling ξ have been tuned as above, that $T^\mu{}_\mu = g^{\mu\nu} T_{\mu\nu} = 0$ if we use the equations of motion of the scalar field.

For the first part of this problem, you may consult Appendix D of R.M. Wald's textbook, *General Relativity*, to find out how the Ricci scalar transforms under local Weyl rescalings.

Students unfamiliar with the concepts of the spacetime metric or the stress-energy tensor might consult:
http://en.wikipedia.org/wiki/Metric_%28general_relativity%29
http://en.wikipedia.org/wiki/Energy-momentum_tensor_%28general_relativity

Problem 5: Given the expression for the Rényi entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\left[\rho_A^n\right] \ ,$$

verify that one recovers the standard von Neumann entropy in the limit $n \to 1$, i.e.,

$$S_1 = \lim_{n\to 1} S_n = -\mathrm{Tr}\left(\rho_A \log \rho_A\right) \ .$$

Verify the same expression for the entanglement entropy emerges from the following limit

$$S_{\mathrm{EE}} = -\lim_{n\to 1} \frac{\partial}{\partial n} \log \mathrm{Tr}\left[\rho_A^n\right] \ .$$

Problem 6: We would like to use the replica trick approach introduced in the first lecture (along with some embelishments appearing in later lectures) to show that in a two-dimensional CFT, the central charge c appearing in the trace anomaly, *i.e.,*

$$\langle T^a{}_a \rangle = \frac{c}{24\pi}\, \mathcal{R} \,,$$

controls the entanglement entropy. In particular, consider the CFT in its vacuum state on a compactified time slice, *i.e.,* on a circle with circumference $2\pi R$, and evaluate the entanglement entropy for *half* of the circle with the following steps: 1) The Renyi entropies are determined with Z_n, a (Euclidean) path integral an n-fold cover of the infinite cylinder. Take advantage of the conformal symmetry to transform this manifold to an n-fold cover of a round two-sphere. 2) Beginning with the formula

$$S_{\mathrm{EE}} = \lim_{\varepsilon\to 0} \left(1 + \frac{\partial}{\partial \varepsilon}\right) \log Z_{1-\varepsilon} \,,$$

relate the derivative $R\partial_R S_{\mathrm{EE}}$ to the conformal anomaly above (and hence the central charge). 3) Integrate the resulting expression over all scales up the final radius R and you should find

$$S_{\mathrm{EE}} = \frac{c}{3} \log\left(R/\delta\right) \ .$$

4) (Optional:) With another Weyl rescaling, one to shift the interval of interest from being half of the initial circle, *i.e.,* the initial time slice, to being an arbitrary interval of angular size 2θ in which case the entanglement entropy becomes

$$S_{\mathrm{EE}} = \frac{c}{3} \log\left(\frac{R}{\delta} \sin\theta\right) = \frac{c}{3} \log\left(\frac{L}{2\pi\delta} \sin\left(\frac{\pi\ell}{L}\right)\right) \ ,$$

where the last expression is a standard result by Calabrese and Cardy [4], with L and ℓ being the circumference of the circle and the length of the interval, respectively. Note that this expression is symmetric under the $\ell \to L-\ell$ and maximal for $\ell = L/2$.

(Optional:) Consider extending the above calculation to a d-dimensional CFT in its vacuum state on $R \times S^{d-1}$, with even d. In particular, let us choose the region A to be the ball on the $t_E = 0$ slice enclosed by $(d-2)$-dimensional sphere with an angular width of θ_0. Verify that the universal contribution to the entanglement entropy is given by

$$S_{\text{univ}} = (-)^{\frac{d}{2}-1} 4\, A\, \log\left(\frac{R}{\delta}\, \sin\theta_0\right) \ ,$$

where A is the central charge appearing in front of the Euler density in the trace anomaly, *i.e.*,

$$\langle T^a{}_a \rangle = \sum B_i \, (\text{Weyl invariants})_i - (-)^{d/2} 2\, A \, (\text{Euler density})$$

In solving this problem, you may wish to consult section 5 of [5].

Problem 7: Killing vectors are used describe the symmetries of a geometry. The fancy way of describing such a symmetry is to say that the Lie derivative of the metric with respect to a Killing vector vanishes

$$\mathcal{L}_k g_{ab} = \nabla_a k_b + \nabla_b k_a = 0 \ .$$

An equivalent approach is to consider an infinitesimal coordinate transformation $x^a \to \tilde{x}^a = x^a + \varepsilon\, k^a$, where ε is an infinitesimal parameter and k^a is a Killing vector. Then the line element must transform as

$$g_{ab} dx^a dx^b \to \tilde{g}_{ab} d\tilde{x}^a d\tilde{x}^b = g_{ab} d\tilde{x}^a d\tilde{x}^b + O(\varepsilon^2) \ .$$

(This transformation is nicely described on the second webpage below.)

Consider the following three vectors:

$$\begin{aligned} \mathbf{k}_1 &= b\ \partial_x \quad (\text{where } b \text{ is a constant}), \\ \mathbf{k}_2 &= x\ \partial_y - y\ \partial_x \ , \\ \mathbf{k}_3 &= x\ \partial_t + t\ \partial_x \ . \end{aligned}$$

Show that these are each Killing vectors of the flat space metric η_{ab}. For each of these vectors, in the (x,y)- or (t,x)-plane, draw a small vector at various points indicating the direction and magnitude of the corresponding vector. The latter exercise should allow you to see that $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ generate a translation along the x-axis, a rotation in the (x,y)-plane and a boost in the (t,x)-plane, respectively.

If one replaces $t = -i\, t_{\text{E}}$, show that the flat space metric becomes

$$ds_{\text{E}}^2 = dt_{\text{E}}^2 + dx^2 + dy^2 + \cdots$$

and the boost Killing vector becomes $\mathbf{k}_3 = i\, \mathbf{k}_{\text{E},3}$ where

$$\mathbf{k}_{\text{E},3} = x\ \partial_{t_{\text{E}}} - t_{\text{E}}\ \partial_x \ .$$

That is, it becomes a rotation in the (t_{E}, x)-plane.

Students who want to get a few more pointers about Killing vectors might consult:

http://en.wikipedia.org/wiki/Killing_vector_field

http://mathworld.wolfram.com/KillingVectors.html

Problem 8: In one of the later lectures, I introduced the idea of a modular or entanglement Hamiltonian to represent a density matrix as an operator

$$\rho_A = e^{-K}/\mathrm{Tr}(e^{-K}) \ .$$

Now in general, K is a complicated nonlocal operator, however, as discussed in the lecture, it is a relatively simple object when considering the Minkowski vacuum of a QFT reduced to the Rindler wedge. The key to this simplification is that the global state (*i.e.,* the vacuum) and the background geometry (*i.e.,* flat space) are invariant under a Killing vector that leaves the entangling surface invariant (*i.e.,* the boost vector $\mathbf{K}_3$ in the previous problem[1]). In this case, the entanglement Hamiltonian can be written in a more covariant form with

$$K = 2\pi \int_A d\Sigma^\mu \ T_{\mu\nu} \ K^\nu$$

where $d\Sigma^\mu$ is the covariant volume element on the spatial slice that is being integrated over, *e.g.,* $d\Sigma^\mu = d^{d-1}y\sqrt{-g}n^\mu$ where n^μ is a unit vector normal to the surface.

Show that in the case of the Rindler wedge, this expression reduces to the same expression given in class if one integrates over the surface $t = 0$. Given the above covariant form one can consider evaluating this expression on other Cauchy surfaces that end of the same entangling surface. Hence try evaluating this expression for the Rindler wedge but on some other interesting Cauchy surface, *e.g.,* $t = b\,x$ where $|b| < 1$. Show that in fact H is a conserved charge. That is, it will yield the same result when evaluated on any Cauchy surface ending on the same entangling surface. Hint: This follows from conservation of the stress tensor, *i.e.,* $\nabla^\mu T_{\mu\nu} = 0$, and the fact that K^μ is a Killing vector, *i.e.,* $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$.

If the QFT being considered is in fact a conformal Killing vector, then the above simplications also apply when K^μ is a conformal Killing vector satisfying

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = \frac{2}{d}\nabla_\sigma K^\sigma \, g_{\mu\nu} \ .$$

In terms of an infinitesimal coordinate transformation, $x^a \to \tilde{x}^a = x^a + \varepsilon\, K^a$, a conformal Killing vector yields

$$g_{\mu\nu}dx^\mu dx^\nu \to \tilde{g}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu = \lambda(\tilde{x})^2\, g_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu + O(\varepsilon^2) \ .$$

That is, rather than remaining invariant, the metric transforms by a Weyl rescaling. Show that for a CFT and K^μ being a conformal Killing vector, the H given above is a still conserved charge, *i.e.,* it yields the same result when evaluated on any Cauchy surface ending on the same entangling surface. Hint: Use the fact that the stress tensor of a CFT is traceless, *i.e.,* ${T_\mu}^\mu = 0$.

Problem 9: Verify the details of the construction of an entropic c-theorem for RG flows of two-dimensional quantum field theories (after it is discussed in the lectures).

[1] Implicitly, here as in the lecture, the entangling surface was chosen as $x = 0$ in the $t = 0$ time slice.

In flat space in higher dimensions, one could could construct an entangling surface consisting of the two flat parallell walls, e.g., the surfaces $x = \pm\ell$ in a constant time surface. One could again apply the same construction as above for this situation in higher dimensions. What are the implications of the resulting inequality?

The original source for the entropic c-theorem in two dimensions is [6]. A discussion of the the second part of the question appears in [7] — see the discussion beginning around eq. (7.2).

Problem 10: Consider $(d+1)$-dimensional anti-de Sitter space with metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right)$$

which describes the boundary CFT in flat space, i.e., the corresponding boundary metric is just $ds^2_{\rm bdy} = -dt^2 + d\vec{x}^2$. Consider a spherical entangling surface in the boundary theory given by $|\vec{x}|^2 = R^2$ on the $t = 0$ slice. Evaluate the holographic entanglement entropy using the Ryu-Takayanagi formula. In particular, what is the universal contribution to the entanglement entropy?

Hint: Show that the extremal surface in the bulk is given by the "hemisphere": $z^2 + |\vec{x}|^2 = R^2$.

This problem was first discussed in [8].

Problem 11: Consider three-dimensional anti-de Sitter space in global coordinates

$$ds^2 = \frac{dr^2}{\frac{r^2}{L^2}+1} - \left(\frac{r^2}{L^2}+1\right)dt^2 + r^2 d\phi^2\,.$$

which is dual to the vacuum state of a two-dimensional boundary CFT on $R \times S^1$. Note that the corresponding boundary metric becomes $ds^2_{\rm bdy} = -dt^2 + L^2 d\phi^2$, and since the angle ϕ has periodicity 2π, the boundary circle has circumference $2\pi L$. The extremal surfaces on constant time slices in the bulk (here, they correspond to geodesics) can be written as

$$r^2(\theta) = \frac{L^2\cos^2\theta_0}{\sin^2\theta_0 - \sin^2\theta}$$

(Optional: Verify the surfaces described by this equation are extremal.)

What is the minimal radius reached reached by one of these surfaces (*i.e.,* for a particular choice of θ_0)? At what angle θ does this surface reach the asymptotic boundary? What is the size of the corresponding entangling region in the boundary metric?

Evaluate the holographic entanglement entropy for one of these surfaces using the Ryu-Takayanagi formula. Note that as discussed in the lectures, since the extremal surface extends all the way to the asymptotic boundary, the area (*i.e.,* the length) of the surface diverges. We can regulate the result by introducing a radial cut-off at $r = L^2/\delta$ where δ is the short-distance cut-off in the boundary CFT. With this regulated integral, show that you recover the well-known result of Calabrese and Cardy [4], which holds for any two-dimensional CFT (and so it must also hold for a holographic CFT):

$$S(\theta_0) = \frac{c}{3}\,\log\left[\frac{2L}{\delta}\,\sin\theta_0\right]\,.$$

Here, we have use $c = 3L/(2G)$ for the central charge of the boundary CFT.

Problem 12: Consider the following vector in d-dimensional flat space

$$\zeta = R\partial_t - \frac{1}{R}[t^2 + |\vec{x}|^2]\partial_t - \frac{2}{R}t\, x^i\partial_i \ .$$

Show that this vector is a conformal Killing vector in flat space. Describe the orbits of ζ — in particular, show that the entangling surface in the previous problem is left invariant. Construct the modular Hamiltonian for the boundary CFT in the previous problem (in terms of the stress tensor T_{ab}).

Problem 13: Consider the following vector in $(d+1)$-dimensional AdS space

$$\xi = \frac{1}{R}[R^2 - z^2 - t^2 - |\vec{x}|^2]\,\partial_t - \frac{2}{R}t\,[z\partial_z + x^i\partial_i] \ .$$

Show that this vector is a Killing vector in AdS space. Show that ξ reduces to ζ from the previous problem on the boundary of AdS space, *i.e.,* in the limit $z \to 0$. Describe the orbits of ξ — in particular, show that the extremal bulk surface in problem 10 is left invariant.

The vectors considered in the previous two problems and related topics are discussed in [10, 20].

Problem 14: Using holographic techniques, evaluate the Rényi entropies for the case described in problem 7.

The solution of this problem may be found in [11].

Problem 15: Examine the details of the Lewkowycz and Maldacena [21] argument proving the Ryu-Takayanagi prescription for holographic entanglement entropy. In particular, show that the location of the singular bulk surface e_n is determined by the equation $K^i = 0$, where K^i is the trace of the two extrinsic curvatures of e_n. (Note that $K^t = 0$ automatically for the static or time-symmetric backgrounds under consideration.) Also show that evaluating the gravity action yields

$$\partial_n I_{grav}(\hat{\mathcal{M}}_n)\Big|_{n=1} = \frac{\mathcal{A}}{4G} \, ,$$

where $\mathcal{A}$ is the area of the extremal surface in the bulk.

You may find the discussion in [22] useful, but you will also find that there are a few typos there.

Problem 16: In the lectures, I mentioned that the F-theorem was originally proved in terms of the entanglement entropy of circles but that a more rigorous proof can be constructed in terms of the mutual information of two circular regions — see [20]. Now at a fixed point, the underlying theory is a CFT and so conformal transformations can be used to map the two concentric circles to other kinds of planar geometries, for which when properly interpreted, the mutual information should yield precisely the same constant. Hence the problem here is to reformulate the proof of the F-theorem discussed in [20] in terms of the mutual information of two regions with a new geometry. In particular, can you find a geometry which may lend itself to a proof that can be extended to higher dimensions?

For students interested in studying topics discussed in my lectures in more detail or more generally, topics related to the new connections between quantum information and quantum gravity, let me suggest that you look at the following:

http://www.perimeterinstitute.ca/it-qubit-summer-school/it-qubit-summer-school-resources

This webpage has a compilation of all of the lectures and problem sets given at the "It from Qubit" Summer School, which was held at the Perimeter Institute, July 18–29, 2016.

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