Some preliminary note

- This is an odd course. It is about AdS/CFT [a too developed topic]. Five lectures cannot make justice

- I could go with some basics. But last week you had a five lectures very nice first course by Diego Tronconelli.

- But the dutch side of the audience did not love that course [but probably loved some other?].

- I was asked by the organisers to discuss specific models and calculations

- The audience is heterogeneous → Cosmology / → Beyond the SM / → AdS/CFT.
  So I need to "cater" for all of them.

- So: I will start with a quite old thing today but is a very powerful idea: Duality.

I stole from many sources to write this lecture on Duality. See my CP3 origins video lecture [go to CP3 origins, they have a CP3 Tube, write my name] There, I gave references and details.

I also used an old set of papers by

Burgess and Quevedo 9401105
and associated lectures by Quevedo 9706210

| Polchinski December 2014 SciPost Rev. Mod Physics

– You will see a long set of problems.

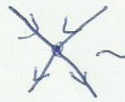
I think that doing one or two a day is enough.

Swieca, 2015

Austen Lobo

Consider a typical QFT in your course.

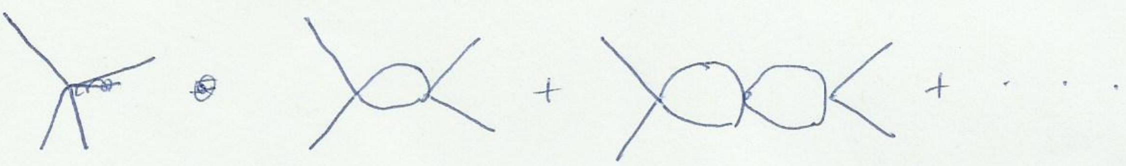
$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{24}\phi^4 \longrightarrow \text{self interacting scalar}$$

 $\sim\lambda$

a given process (or the propagator, for example)

$\lambda^0 \qquad \lambda \qquad \lambda^2 \qquad + \quad \lambda^2 + \dots \lambda^3 + \dots$

if λ is small it is good to calculate a couple of diagrams (modulo renormalizations).



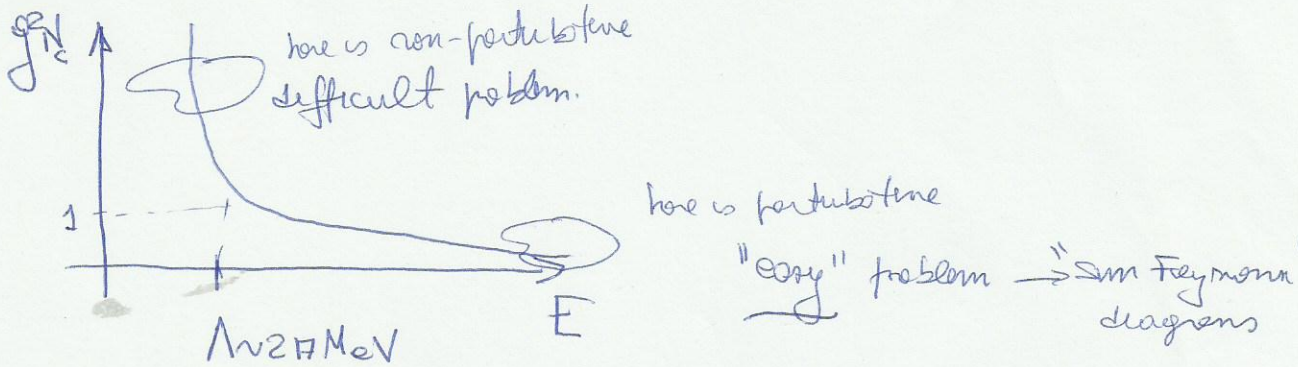
$\lambda \; + \; \lambda^2 \; + \; \lambda^3 \; + \; \dots$

If, on the other hand, λ is $\underline{\lambda \geq 1}$ calculating diagrams is not a way to go.

$\leadsto$ non-perturbative problem $\longrightarrow$ is this interesting?

Are non-perturbative problems interesting in Physics?

Yes; for example consider QCD



Is the non perturbative problem of interest?

Yes: formation of proton

" " pion and other particles, etc

So, what do we do? People have developed

different approaches

- Lattice : in the case of YM and QCD → best.
  but with limitations as soon as you start
  changing the theory slightly.

- Some QFT elaboration : Schwinger-Dyson eq. etc [not much Instantons]
  Instantons

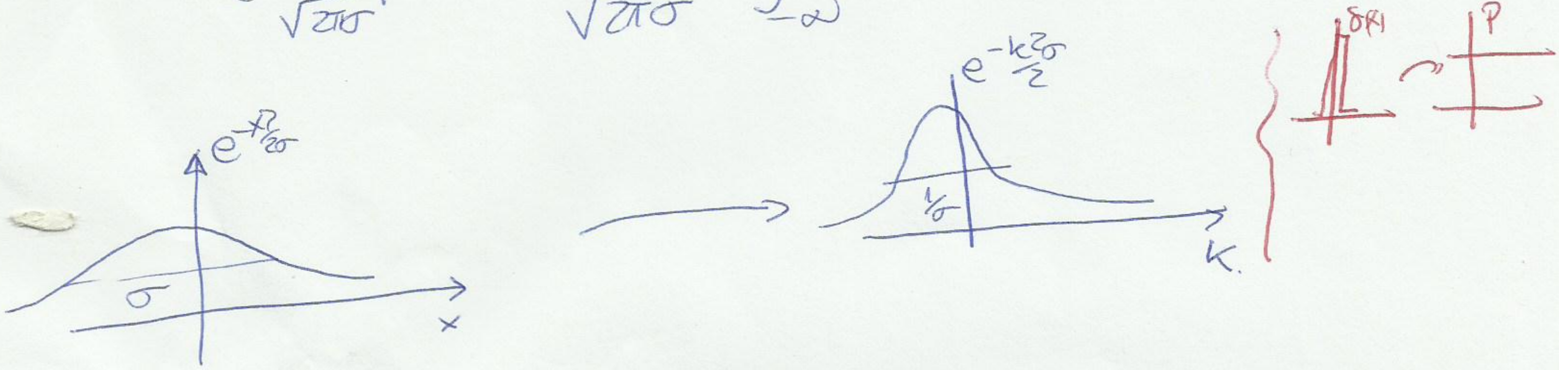
- Duality (of which "effective field theory" is a particular case)

What will I mean by "Duality"?

Let me explain this with examples

Example 1 : Fourier transform and uncertainty principle

$$\mathcal{F}\left(\frac{e^{-x^2/2\sigma}}{\sqrt{2\pi\sigma}}\right) = \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} dx\, e^{ikx} e^{-x^2/2\sigma} = e^{-\frac{k^2\sigma}{2}}$$



Example 2 : Maxwell eqs. (in vacuum) $\bar{J}=0$, $\rho=0$

$$\nabla\cdot\bar{E}=0 \qquad \nabla\cdot\bar{B}=0$$
$$\nabla\times\bar{B}=+\partial_t\bar{E} \qquad \nabla\times\bar{E}=-\partial_t\bar{B}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\partial_\mu F^{\mu\nu} = J^\nu \to 0 \qquad \partial_{[\mu} F_{\nu\rho]} = 0$$

$$dF=0$$

we see their invariance of the exchange

$$\bar{E} \longrightarrow -\bar{B}$$
$$\bar{B} \longrightarrow \bar{E}$$

$$\wr$$

$$F_{\mu\nu} \to \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$\boxed{F_{\mu\nu} \longrightarrow {}^*F_{\mu\nu}}$$

Example 3: Bosonisation (in 1+1d) [Coleman+Mandelstam (1975)]

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 + \frac{\alpha}{\beta^2}\cos(\beta\phi) \xleftrightarrow{\text{Same}} \mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi + \frac{g}{2}(\bar{\Psi}\gamma_\mu\Psi)^2$$

$$J_\mu = \varepsilon_{\mu\nu}\partial_\nu\phi \qquad\qquad J_\mu = \bar{\Psi}\gamma_\mu\Psi$$

Order / disorder operators

how to interpret the new variables σ_i ?

Calculamos $\langle S_i \rangle$; $\langle \sigma_i \rangle$

$$\langle S_i \rangle = \lim_{h_i \to 0} \frac{d}{dh_i} \log Z[h]$$

$$\langle S_i \rangle = \frac{1}{Z[0]} \frac{d}{dh} Z[h] \Big|_{h=0}$$

$$\boxed{Z[h] = \sum_{\{S\}} e^{\beta\left(\sum_{\langle ij \rangle} s_i s_j + \sum h_i s_i\right)}}$$

for $\beta < \beta_c$ low temperatures

$\beta > \beta_c$ high temperature

$\langle S_i \rangle = 1$ $\longleftrightarrow$ $\langle S_i \rangle = 0$

$\langle \sigma_i \rangle = 0$ $\qquad$ $\langle \sigma_i \rangle = 1$

No simple one to one correspondence between $\{S_i\} \longrightarrow \{\sigma_i\}$

$$\frac{4\pi}{\beta^2} \longrightarrow 1 + \frac{g}{\pi}$$

$$e^{\int \phi\, d^2x}_{\varepsilon} \longrightarrow \psi$$

Example 4 Ising Model in 1+1 (Kramers + Wannier 1940)

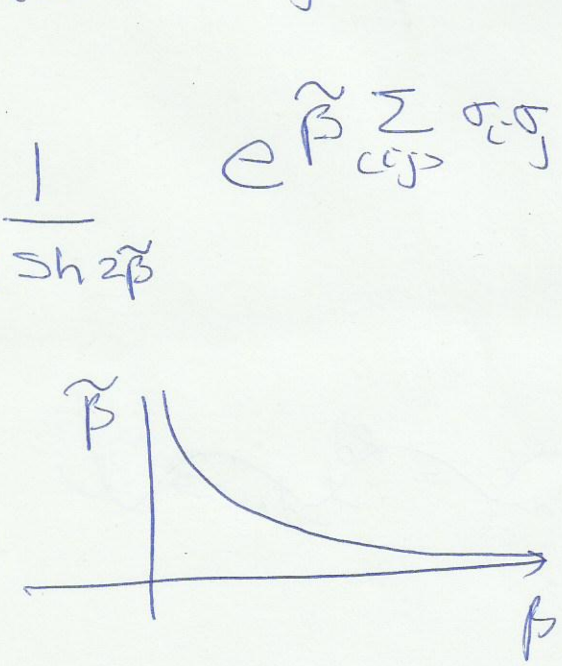
spins on a line $\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow \cdots$

$$H = -J \underbrace{\sum_{<ij>}}_{\text{nearest neighbours.}} S_i S_j$$

$$Z = \sum_{\{S\}} e^{\beta \sum_{<>} S_i S_j} \qquad \beta = -J/kT$$

one can show that this infinite sum can be written as

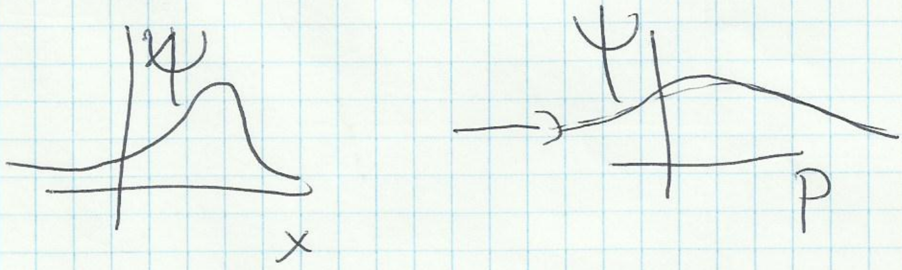
$$Z = \sum_{\{\sigma\}} \frac{1}{2\,\mathrm{sh}\, 2\tilde{\beta}} \, e^{\tilde{\beta} \sum_{<ij>} \sigma_i \sigma_j} \qquad \text{where } e^{2\tilde{\beta}} = \coth\beta$$



- Like in usual Fourier transform the configurations of the "disorder" operator are <u>Sums</u> of configurations of "order" operators.

- When the "order" operators are very fluctuating (high temperature) the "disorder" operators are almost classical

Same as Fourier



So, there are some "characteristic" things for all these examples

- Exchange eqs of motion $\longleftrightarrow$ Bianchi identities
- preserve <u>some</u> global symmetries
- Exchange weak and strong "coupling"
- Degrees of freedom on both dual descriptions are quite different.
- Actually one set of degrees of freedom is a <u>superposition</u> of all the others degrees of freedom in the dual description

(Fourier / Ising / Bosonization)

- When one set of degrees of freedom is <u>very fluctuating</u> (Quantum Mechanics) the dual set is "almost classical".

So, very quantum Mechanical problem

(like a non-perturbative problem) turns

semiclassical in a given dual description.

→ hence the use of duality!

There are many other dualities (in diverse dimensions

and using different degrees of freedom/theories) they all

follow similar patterns. and the previous points

Basics on Path Integral

Suppose a Lagrangian $\mathcal{L}(\phi, A_\mu, \psi)$
the quantum Mechanics of the system is
described by

$$Z[J_1, J_\mu, \eta] = \int D\phi D A_\mu D\bar{\psi} D\psi \, e^{-\frac{i}{\hbar}\int \mathcal{L} d^4x + J_1 \mathcal{O}(\phi) + J_\mu \mathcal{O}^\mu_{(A_\mu)} + \eta\, \mathcal{O}(\psi)}$$

that also calculates partition functions $\rightarrow$ <u>observables</u>

A typical trick to impose a condition is using a
Lagrange multiplier [a field without dynamics]

$$\mathcal{L} = \left[\mathcal{L}[\phi, A_\mu, \psi, \bar{\psi}] + \Lambda(\phi^2 - 1)\right]$$ this imposes $\phi^2 = 1$

Since $\dfrac{\partial \mathcal{L}}{\partial \Lambda} = \underline{\phi^2 - 1 = 0}$ ✓.

Many times, Path Integrals can be explicitly evaluated
(quadratic actions) $\rightsquigarrow$ "free problems"

$$\boxed{\hat{\mathcal{O}} = \Box + m^2}$$

$$\int D\phi \, e^{-i\int \frac{1}{2}\phi \hat{\mathcal{O}} \phi} \sim \left[\det \hat{\mathcal{O}}\right]^{\phi}$$

Duality "procedure"

- detect a global symmetry
- gauge it, but impose the gauge field is not propagating using a Lagrange multiplier
- Integrate "the other way around"
  ~> that is the dual theory.

Example: Bosonization
- T-duality
- Ising model

$d = 1+1$ Bosonization (Abelian).

Suppose that we start with a ~~[illegible]~~ globally symmetric fermionic (Dirac) theory

$$\mathcal{L} = i \bar{\Psi} \not\partial \Psi.$$

and we add an external current

$$\mathcal{L}[s] = i \bar{\Psi}(\not\partial + \not s)\Psi$$

$$\rightarrow \quad Z[s_\mu] = \int D\bar{\Psi} D\Psi \, \exp\left[i \int d^2x \, \bar{\Psi}(\not\partial + \not s)\Psi \right]$$

- the Lagrangian and the $Z[s_\mu]$ have a global symmetry.

$$\Psi \to e^{i\alpha}\Psi$$
$$\bar{\Psi} \to \bar{\Psi} e^{-i\alpha}$$

- we gauge that symmetry.

$$Z[s_\mu] = \int D\bar{\Psi} D\Psi \, DA_\mu \, \exp\left[i \int d^2x \, \bar{\Psi}(\not\partial + \not A + \not s)\Psi \right]$$

- but we impose no dynamics for A_μ $\longrightarrow \delta(F_{\mu\nu} = 0)$

We rewrite it in this way (and we gauge fix)

$$Z[S_\mu] = \int D\bar{\psi}\, D\psi\, DA_\mu\, D\Lambda \;\times$$

$$\exp\left\{ i \int d^2x\; \bar{\psi}\,(\partial\!\!\!/ + A\!\!\!/ + S\!\!\!/)\,\psi + \frac{\Lambda}{2}(\varepsilon_{\mu\nu}F_{\mu\nu}) \right\}$$

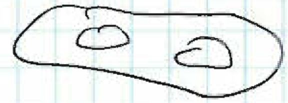
$\cdot\,\delta(\partial_\mu A^\mu)$

↳ gauge fixing term

- So, if we integrate over Λ $\rightsquigarrow F_{\mu\nu}=0 \to A_\mu = \partial_\mu \chi$

together with the gauge condition $\partial_\mu A^\mu = 0$

implies $\chi = \text{const} \rightsquigarrow A_\mu = 0$ [in spaces with nontrivial topology this may not be the case]



But the dual version is obtained by integrating

- fermions
- gauge field

→ giving an action for "Λ"

$$e^{-S[\Lambda, s]} = \int DA_\mu D\bar{\psi} D\psi \; e^{\left[-i\int d^2x \; \bar{\psi}(\partial\!\!\!/ + A\!\!\!/ + s\!\!\!/)\psi + \frac{\Lambda}{2}\varepsilon_{\mu\nu}F_{\mu\nu}\right]} \delta(\partial_\mu A^\mu)$$

$$* \int D\bar{\psi} D\psi = \det(\partial\!\!\!/ + A\!\!\!/ + s\!\!\!/) \quad \text{(Schwinger 1962)}$$

$$= e^{\frac{i}{4\pi}\int d^2x \; F_{\mu\nu}\frac{1}{\Box}F_{\mu\nu}} \longleftarrow \left(\text{allow me to write } s_\mu = 0\right)$$

So we have

$$\int DA_\mu \; e^{i\int d^2x \frac{1}{4\pi} F_{\mu\nu}\frac{1}{\Box}F_{\mu\nu} + \frac{\Lambda}{2}\varepsilon_{\mu\nu}F_{\mu\nu}} \cdot \delta(\partial_\mu A^\mu)$$

Now, we make a change of variables

$$A_\mu = \partial_\mu \chi + \varepsilon_{\mu\nu}\partial_\nu \varphi$$

in the gauge $\partial_\mu A_\mu = 0 \rightsquigarrow \Box\chi = 0 \longrightarrow \chi = \text{constant}$ (Theorem of functions ... solutions)

$$\rightsquigarrow A_\mu = \varepsilon_{\mu\alpha}\partial_\alpha \varphi$$

$$DA_\mu \rightsquigarrow J\, D\varphi \qquad J = \frac{\delta A_\mu}{\delta\varphi} = \det(\varepsilon_{\mu\nu}\partial_\nu)$$

$$\left[F_{\mu\nu} = \varepsilon_{\nu\alpha}\partial_\mu\partial_\alpha\varphi - \varepsilon_{\mu\alpha}\partial_\nu\partial_\alpha\varphi\right]$$

$$\int D\varphi \; \exp\left\{-i\int \frac{F_{\mu\nu}}{4\pi}\frac{1}{\Box}F_{\mu\nu} + \frac{\Lambda}{2}\varepsilon_{\mu\nu}F_{\mu\nu}\right\}$$

$$A_\mu = \varepsilon_{\mu\alpha} \partial_\alpha \varphi$$

$$A_\nu = \varepsilon_{\nu\alpha} \partial_\alpha \varphi \quad \Big] \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = \varepsilon_{\nu\alpha} \partial_\mu \partial_\alpha \varphi - \varepsilon_{\mu\alpha} \partial_\nu \partial_\alpha \varphi$$

$$F_{\mu\nu} \frac{1}{\Box} F^{\mu\nu} = \varepsilon_{\nu\alpha} \partial_\mu \partial_\alpha \varphi \, \varepsilon_{\nu\beta} \partial_\mu \partial_\beta \varphi - 2 \varepsilon_{\nu\alpha} \varepsilon_{\mu\beta} \partial_\mu \partial_\alpha \varphi \partial_\nu \partial_\beta \varphi$$
$$+ \varepsilon_{\mu\alpha} \varepsilon_{\mu\beta} \partial_\nu \partial_\alpha \varphi \, \partial_\nu \partial_\beta \varphi$$

$$\varepsilon_{\mu\nu} F_{\mu\nu} = \varepsilon_{\mu\nu} \varepsilon_{\nu\alpha} \partial_\mu \partial_\alpha \varphi - \varepsilon_{\mu\nu} \varepsilon_{\mu\alpha} \partial_\nu \partial_\alpha \varphi$$

$$\left[\frac{\varepsilon_{\mu\nu}}{2} F_{\mu\nu} = \frac{1}{2} \left(\delta_{\mu\alpha} \partial_\mu \partial_\alpha \varphi - (-\delta_{\nu\alpha}) \partial_\nu \partial_\alpha \varphi \right) = \Box \varphi \right]$$

$$F \frac{1}{\Box} F = 2 \varepsilon_{\nu\alpha} \varepsilon_{\nu\beta} \partial_\mu \partial_\alpha \varphi \partial_\mu \partial_\beta \varphi - 2 \varepsilon_{\nu\alpha} \varepsilon_{\mu\beta} \partial_\mu \partial_\alpha \varphi \, \partial_\nu \partial_\beta \varphi$$

$$= 2 \delta_{\alpha\beta} \partial_\mu \partial_\alpha \varphi \partial_\mu \partial_\alpha \varphi - 2 \left(\delta_{\mu\nu} \delta_{\alpha\beta} - \delta_{\nu\beta} \delta_{\alpha\mu} \right) \left(\partial_\mu \partial_\alpha \varphi \right) \left(\partial_\nu \partial_\beta \varphi \right)$$

$$= \cancel{2 (\partial_\mu \partial_\alpha \varphi)(\partial_\mu \partial_\alpha \varphi)} - \cancel{2 (\partial_\mu \partial_\alpha \varphi)(\partial_\mu \partial_\alpha \varphi)} + 2 (\partial_\mu \partial_\mu \varphi)(\partial_\nu \partial_\nu \varphi)$$

$$F \frac{1}{\Box} F = \frac{2}{\Box} \Box \varphi \, \Box \varphi = 2 \varphi \Box \varphi .$$

$$\Rightarrow \int D\varphi \cdot \exp \left[-i \int d^2x \, \frac{2}{2} \varphi \Box \varphi + \Lambda \Box \varphi \right]$$

Integrate by parts $\int D\varphi \exp \left[-i \int d^2x \, \varphi \Box \varphi + \Box \Lambda \varphi \right]$

Now, Integrate φ

introduce S_μ back

$$\int \mathcal{D}\varphi \, e^{(\;)} = e^{-i\int d^2x \, \frac{\pi}{2}\partial_\mu\Lambda\partial_\mu\Lambda + \varepsilon_{\mu\nu}\partial_\nu\Lambda S_\mu}$$

$$\Lambda = \frac{\phi}{\sqrt{\pi}}$$

$\Rightarrow$

$$\boxed{Z[S_\mu] = \int \mathcal{D}\phi \, \exp\left[-i\int d^2x \, (\partial_\mu\phi)^2 + \frac{1}{\sqrt{\pi}} S_\mu \partial_\nu \phi \varepsilon^{\mu\nu}\right]}$$

we see that

$$i\bar{\Psi}\not{\partial}\Psi \longleftrightarrow \frac{1}{2}(\partial_\mu\phi)^2$$

$$g\bar{\Psi}\gamma_\mu\Psi \longleftrightarrow \frac{1}{g}\frac{\varepsilon_{\mu\nu}}{\sqrt{\pi}}\partial_\nu\phi$$

- Notice that $g \longleftrightarrow \frac{1}{g}$
- Noether current $\longrightarrow$ topological current

Conserved thanks to $\not{\partial}\psi = 0$ Fermion $\longrightarrow$ $\underline{\text{Soliton}}$ Conserved

Another way of doing the integrals

Back to the "initial" path Integral

$$Z[s_\mu] = \int D\bar{\psi}\, D\psi\, DA_\mu\, D\Lambda \, \exp\left[-i\int d^2x \; \bar{\Psi}(\partial\!\!\!/ + A\!\!\!/ + s\!\!\!/)\psi + \frac{\Lambda}{2}\varepsilon_{\mu\nu}F_{\mu\nu}\right] \cdot \delta(\partial_\mu A^\mu)$$

re-write it as

$$Z[s_\mu] = \int D\bar{\psi} D\psi\, DA_\mu\, D\Lambda\, D\omega$$

$$\exp\left[-i\int d^2x \; \bar{\Psi}(\partial\!\!\!/ + A\!\!\!/ + s\!\!\!/)\psi + \underbrace{\frac{\Lambda}{2}\varepsilon_{\mu\nu}F_{\mu\nu}}_{-A_\mu \varepsilon_{\mu\nu}\partial_\nu \Lambda} + \underbrace{\omega\, \partial_\mu A^\mu}_{-A^\mu \partial_\mu \omega}\right]$$

$$\int D\text{fields} \times$$

$$= \exp\left[-i\int d^2x \; \bar{\Psi}(\partial\!\!\!/ + s\!\!\!/)\psi + A_\mu\left(\bar{\Psi}\gamma^\mu\psi - \varepsilon_{\mu\nu}\partial_\nu\Lambda - \partial_\mu\omega\right)\right]$$

$\Rightarrow$ Integrate over A_μ

$$Z[s_\mu] = \int D\bar{\psi} D\psi D\Lambda D\omega \; e^{-i\int d^2x \; \bar{\Psi}(\partial\!\!\!/ + s\!\!\!/)\psi} \cdot \delta(\bar{\Psi}\gamma^\mu\psi - \varepsilon_{\mu\nu}\partial_\nu\Lambda - \partial_\mu\omega)$$

So, we have

$$Z[S_\mu] = \int D\bar\Psi D\Psi\, D\Lambda\, D\omega\; e^{-i\int d^2x\, \bar\Psi(\not\partial + \not S)\Psi} \cdot \delta\left(\bar\Psi\gamma^\mu\Psi - \epsilon_{\mu\nu}\partial_\nu\Lambda - \partial_\mu\omega\right)$$

how to calculate this path integral?

El punto es considerar an axial symmetry

$\bar\Psi\gamma^\mu\gamma_5\Psi$ que se acopla a $S_\mu \to b_\mu + \overset{\text{axial vector}}{S_\mu}$

bajo una axial symmetry

$$\bar\Psi \to \bar\Psi e^{i\alpha\gamma_5}$$
$$\Psi \to e^{i\alpha(x)\gamma_5}\Psi$$
$$b_\mu \to b_\mu + \partial_\mu\alpha$$

esta simetría es anómala $\rightsquigarrow$

$$i\bar\Psi\gamma_\mu\gamma_5\Psi \to i\bar\Psi\gamma_\mu\gamma_5\Psi + \frac{1}{\pi}\partial_\mu\alpha$$

$$i\bar\Psi\gamma_\mu\Psi \to i\bar\Psi\gamma_\mu\Psi - \frac{1}{\pi}\epsilon_{\mu\nu}\partial_\nu\alpha$$

El Lagrangiano

$$\mathcal{L}_F = i\bar\Psi(\not\partial + i\gamma_\mu A_\mu + i\gamma_\mu\gamma_5 b_\mu)\Psi \quad \text{transforma según}$$

$$\mathcal{L} = \mathcal{L}_F + \frac{1}{\pi}\partial_\mu\alpha\,(b_\mu + \epsilon_{\mu\nu}A_\nu)$$

El termino $\partial_\mu \alpha \, \varepsilon_{\mu\nu} A_\nu$ es la axial anomaly!

Esta anomalia se puede cancelar si uno impone

$$\Lambda \longrightarrow \Lambda - \frac{\alpha}{\pi}$$

ya que la combinacion

$$\bar{\Psi} \gamma^\mu \Psi - \varepsilon_{\mu\nu}\partial_\nu \Lambda - \partial_\mu \omega \rightsquigarrow$$

$$\bar{\Psi} \gamma^\mu \Psi - \frac{1}{\pi} \varepsilon_{\mu\nu}\partial_\nu \alpha - \varepsilon_{\mu\nu}\partial_\nu \Lambda + \frac{\varepsilon_{\mu\nu}}{\pi}\partial_\nu \alpha - \partial_\mu \omega .$$

$\rightsquigarrow$ el Lagrange multiplier es basicamente via el termino $\Lambda\, \varepsilon_{\mu\nu} F_{\mu\nu}$ el que al cambiar $\Lambda \to \Lambda - \frac{\alpha}{\pi}$ da cuenta de la anomalia axial! $\rightsquigarrow$ Λ esta haciendo un Green-Schwartz Mechanism (WZ term)

$\rightsquigarrow$ el Lagrangiano es / $\mathcal{L}(\Lambda, b) - \frac{1}{2} b_\mu^2$ es invariante

$\rightsquigarrow$ $$\mathcal{L} = -\frac{\pi}{2}\left(\partial_\mu \Lambda - \frac{b_\mu}{\pi} - \varepsilon_{\mu\nu}\partial_\nu \Lambda\, a_\mu\right)^2 + \frac{b_\mu^2}{2\pi}$$