1 Problems for Lecture 1

1- Given the background for D3 branes,

$$ds^2 = h[r]^{-1/2} dx_{1,3}^2 + h[r]^{1/2}[dr^2 + r^2 d\Omega_5], \tag{1.1}$$
$$A_{tx_1x_2x_3}(r) = -\frac{1}{2}(h[r]^{-1} - 1).$$
$$F_5 = F_{tx_1x_2x_3x_4r} + F_{\theta_1\theta_2\theta_3\theta_4\theta_5} = dA_4(1 + *_{10}).$$
$$h[r] = 1 + \frac{4\pi g_s N_c \alpha'^2}{r^4}.$$

Here, α' is roughly the square of the string lenght, N_c and g_s are two parameters of the string theory.

For the moment, let us focus on the metric, we will deal with the Ramond Field F_5 later.

-What is the asymptotic behaviour of this background? I mean what happens to the metric, for $r \to 0$ and $r \to \infty$?

-Let us follow Maldacena's idea and define a new coordinate

$$u = \frac{r}{\alpha'} \tag{1.2}$$

and, rewrite the background above.

-Now, take the limit $\alpha' \to 0$, keeping fixed u (we will keep also $g_s N_c$ fixed, in the limit of $g_s \to 0, N_c \to \infty$). The background should look like

$$ds^2 = \alpha'\Big[\frac{u^2}{L^2} dx_{1,3}^2 + \frac{L^2}{u^2} du^2 + L^2 d\Omega_5\Big]. \tag{1.3}$$

If this is the case, do you obtain that $L^4 = g_s N_c$?

-What are the units of the radial coordinate u?

-Please write explicitly the F_5. Discuss with a classmate what we mean by the self dual $*_{10}$. This is a technicality (somewhat useful to know if you work on AdS/CFT), so if you feel like covering it, go ahead. Discuss with classmates and the tutors [they have the solutions]. If not, talk to me.

2-

We wrote AdS-space [in five dimensions] as

$$ds^2 \sim u^2 dx_{1,3}^2 + \frac{du^2}{u^2} \tag{1.4}$$

Where is the 'UV' and the 'IR 'of the QFT dual in terms of the u coordinate? (of course, it is a CFT, there is no scale to compare with)

Now, I will write AdS five dimensional space in another systems of coordinates. Can you please, find the change of variables from one to the other?

$$ds^2 \sim \frac{1}{z^2}(dz^2 + dx_{1,3}^2)$$
$$ds^2 \sim e^{2r} dx_{1,3}^2 + dr^2.$$

Can you tell in each coordinate where is the 'UV' and the 'IR' of the dual QFT?

3- All right, now, let us do the same for the cases of D2 and D4 branes. The general background for a Dp brane is (there is no self-duality condition in these cases),

$$ds_{st}^2 = h^{-1/2} dx_{1,p}^2 + h^{1/2}[dr^2 + r^2 d\Omega_{8-p}], \tag{1.5}$$
$$A_{tx_1x_2x_3\ldots x_p}(r) = -\frac{1}{2}(h^{-1} - 1)$$
$$F_{p+2} = dA_{p+1}. \quad e^{-4\phi+4\phi_0} = h^{p-3}.$$
$$h(r) = 1 + \frac{(2\pi)^{p-2} d_p g_s N_c (\alpha')^{\frac{7-p}{2}}}{r^{7-p}}.$$
$$d_p = 2^{7-2p}\pi^{(9-p)/2}\Gamma(\frac{7-p}{2}). \quad g_{YM}^2 = (2\pi)^{p-2} g_s (\alpha')^{(p-3)/2}. \tag{1.6}$$

A paper that I always found quite inspiring on this, was written by Itzhaki, Maldacena, Sonnenschein and Yankielowicz hep-th/9802042. Also, take a look at the paper by Boonstra, Skenderis and Townsend hep-th/9807137, that presents a different view on the same systems. If you find yourself confused, we can talk about it.

4- Now, we go for something different. We will discuss two examples of 'Duality'. The aim of these two problems is to show you something that we tangentially discussed in the first lecture and that is not quite discussed in these days—but is very nice. I will be quite detailed, some of you will not need so much guidance, if so, help others.

a- A good reference for this problem is the paper hep-th/9401105. By Burgess and Quevedo

Consider a two dimensional QFT of free fermions. The partitions function reads,

$$Z = \int D\bar{\psi} D\psi e^{-\int d^2x L}, \quad L = i\bar{\psi}\partial\psi \tag{1.7}$$

with the symbol $\partial = \gamma^\mu \partial_\mu$, as you know.

You observe that there is an invariance in this Action

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\alpha}. \tag{1.8}$$

Suppose now that we want to 'gauge this invariance'. You know how to do that. But we want also to impose that the gauge field introduced, has no degrees of freedom (that is pure gauge, in other words). We can do that by introducing a Lagrange multiplier Λ that will impose that the curvature of the gauge field vanish $F_{\mu\nu} = 0$.

If we do so, the partition function will read,

$$Z = \int D\bar{\psi} D\psi DA_\mu D\Lambda \exp\Big[\int d^2x (i\bar{\psi}[\partial + A]\psi + \frac{\Lambda}{2}\epsilon_{\mu\nu}F^{\mu\nu})\Big]. \tag{1.9}$$

where above, $A = \gamma^\mu A_\mu$. We could also add external currents, but let us not do it here.

–Now, i ask you check that if you first integrate over Λ and then over A_μ, you recover the partition function of eq.(1.7).

–Now, we will do something more interesting. We will first integrate over the fermions. Please, use that

$$Z = \int D\bar{\psi} D\psi e^{\int d^2x \bar{\psi}(\partial + A)\psi} = \exp \frac{i}{4\pi} \int d^2x \frac{F^{\mu\nu}F_{\mu\nu}}{\nabla^2}, \tag{1.10}$$

this one above is an exact result in two dimensions only.

–Now your partition function should read,

$$Z = \int DA_\mu d\Lambda \exp i \int d^2x \Big[\frac{F_{\mu\nu}F^{\mu\nu}}{4\pi\nabla^2} + \frac{\Lambda}{2}\epsilon_{\mu\nu}F^{\mu\nu}\Big] \tag{1.11}$$

Now, one can integrate over A_μ. To do this, it is convenient to change variables,

$$A_\mu = \epsilon_{\mu\nu}\partial_\nu \Phi. \tag{1.12}$$

This change of variables involves a trivial Jacobian, so you can check that the integration over Φ (that is the A_μ integration) leads you to, a Lagrangian,

$$L = -\frac{\pi}{2}(\partial_\mu \Lambda)^2. \tag{1.13}$$

If you followed all these steps, let us recapitulate: you started from a system for fermions $(\psi, \bar{\psi})$. You did some operations and obtained a system in terms of a boson (Λ). This is called bosonisation.

The idea is very far reaching, has many interesting sides and applications. Talk to me if you are interested.

b-

Consider the 4-d field theory with partition function,

$$Z[J] = \int DA_\mu \exp\Big[-\frac{i}{4g^2}\int d^4x F_{\mu\nu}^2 + \int d^4x J_{\mu\nu}F^{\mu\nu}\Big]\delta(\partial_\mu A^\mu). \tag{1.14}$$

This describes an abelian gauge field, coupled to an external current J_2. There is a gauge fixing term. One can calculate correlators of field strenghts by doing successive derivatives respect to the external current. Please, write a formal expression for

$$< F_{\mu\nu}(x) F_{\alpha\beta}(y) > \tag{1.15}$$

in terms of derivatives of the previous partition function.

– Now we will change variables, instead of integrating over A_μ, we will integrate over $F_{\mu\nu}$. Convice your classmates that the good way of doing this is by writing

$$Z[J] = \int DF_{\mu\nu} \exp\Big[-\frac{i}{4g^2}\int d^4x F_{\mu\nu}^2 + \int d^4x J_{\mu\nu}F^{\mu\nu}\Big]\delta(\epsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta}). \tag{1.16}$$

What is the role played by the delta function term?

— What happens with the Jacobian in moving from $A_\mu \to F_{\mu\nu}$ integration?

—- Now, argue that you can write the last partition function as,

$$Z[J] = \int DF_{\alpha\beta}DC_\mu \exp\Big[-\frac{i}{4g^2}\int d^4x F_{\mu\nu}^2 + \int d^4x J_{\mu\nu}F^{\mu\nu} + \frac{1}{4\pi}\int d^4x \epsilon^{\mu\nu\alpha\beta}C_\mu\partial_\nu F_{\alpha\beta}\Big]\delta(\partial_\mu C^\mu). \tag{1.17}$$

We wrote the last delta function to 'fix the gauge invariance' $C_\mu \to C_\mu + \partial_\mu p$ (p is some function). Is it clear that such invariance is there?

—— Good, now let us integrate by parts and write the term,

$$\int d^4x \epsilon^{\mu\nu\alpha\beta}C_\mu\partial_\nu F_{\alpha\beta} = -\frac{1}{2}\int d^4x \epsilon^{\mu\nu\alpha\beta}f_{\mu\nu}F_{\alpha\beta}. \tag{1.18}$$

——– Now, since the Action is quadratic in $F_{\mu\nu}$, let us integrate over $F_{\mu\nu}$. You should get something along the lines of,

$$Z[J] = e^{-ig^2\int J^2 d^4x}\int Df_{\alpha\beta} \exp\Big[-\frac{ig^2}{64\pi^2}\int d^4x f_{\mu\nu}^2 + g^2\int d^4x J_{\mu\nu}f^{\mu\nu}\Big]\delta(\partial_\mu C^\mu). \tag{1.19}$$

Do we roughly agree? The first prefactor is a contact term. The important thing here is to notice that the 'coupling gets inverted' [of course this theory is not interacting!] and that the original variables A_μ are related in a very complicated form to the new variable C_μ. But this a Maxwell action into another Maxwell action. **This is a toy example.** There are more elaborated versions of this argument, for example what is called Montonen-Olive duality.

If after this, you feel curious to see this at work in clearer and far more useful examples, take a look at the beautiful papers by Burgess and Quevedo hep-th/9401105 and hep-th/9403173.