Lecture 3
Summary

AdS/CFT $\qquad \overbrace{\mathcal{N}=4\ SYM}^{\text{D3 brns}} \longrightarrow \overbrace{AdS_5\times S^5}^{\text{D3 brns}}$

- This generalises to other Dp brans
  without "AdS" without "CFT"

- $$ds^2_{AdS} \sim u^2\, dx_{1,3} + \frac{du^2}{u^2}$$

$U \sim$ Energy coordinate $\qquad u\to\infty$ UV
$\qquad\qquad\qquad\qquad\qquad\qquad u\to 0$ IR

- $$R_{AdS} = R_{S^5} \quad ; \quad \boxed{\frac{R^4}{\alpha'^2} = \lambda_{\cancel{t'Hooft}}}$$

keep R fixed $\longrightarrow \alpha' \sim \dfrac{1}{\sqrt{\lambda}}$ $\qquad$ $\underline{\text{classical string}}$
α' corrections $\sim \dfrac{1}{\sqrt{g^2N}}$ corrections

$(2\kappa_{10}^2 = g_s^2\alpha'^4(2\pi)^7)$

$$\frac{2\kappa_{10}^2}{\alpha'^4} = g_s^2(2\pi)^7 \sim \frac{\lambda_t^2}{N^2} \longrightarrow g_s \text{ corrections} \sim \frac{1}{N^2}$$

$\underline{\text{Quantum gravity}}$ corrections

$AdS_5 \times S^5$ has a boundary

$$ds^2 \sim u^2 \, dx_{1,3}^2 + \frac{du^2}{u^2} + d\Omega_5$$

$$\sim u^2 \left[dx_{1,3}^2 + \frac{du^2}{u^4} + \frac{1}{u^2} d\Omega_5 \right]$$

$u \to \infty$

$$\underline{ds^2_{\text{boundy}} \sim dx_{1,3}^2}$$

Studying fields in AdS $\longrightarrow$ Boundary

Condition. $\longrightarrow$ <u>Exercise</u>

Operative version of the Maldacena Conjecture

for a QFT person

$$Z_{N=4\,SYM}[J_i] = \int DA_\mu^a D\vec{\phi}^a D\lambda^a Dghost \; \exp\left[-\int \mathcal{L}_{N=4\,SYM}\, d^4x + \int J_i\, \mathcal{O}^i\right]$$

is the quantity of Interest.

The proposal by Gubser-Klebanov-Polyakov (1998)
Witten

is that

conjecture

$$Z_{N=4\,SYM}[J_i] \equiv Z_{\text{IIB String theory on } AdS_5\times S^5 \text{ with boundary conds } J_i}\left[\Phi_i \xrightarrow{\text{boundary}} J_i\right]$$

for an string theorist studying string on $AdS_5 \times S^5$

$$Z = \int \overset{\text{String fields}}{D\Phi_i}\, e^{-S_{IIB}} \Big|_{\substack{\text{bc that} \\ \Phi_i \to J_i \text{ on boundary.}}}$$

Boundary of $AdS_5 \times S^5$

$$ds^2 = \frac{\alpha' u^2}{R}\left\{ \frac{dx_{1,3}^2}{R^2} + \frac{R^2 du^2}{u^4} + \frac{R^2}{u^2} d\Omega_5 \right\}$$

$$\lim_{u \to \infty} ds^2 = dx_{1,3}^2$$

Both $Z_{N=4\,SYM}[J_i]$ and $Z_{IIB\,String}[\Phi_i \to J_i]$ are very difficult to compute. But if we are for example in the semiclassical $g_s \to 0$ or point particle limit $\alpha' \to 0$ of the string one may approximate conjecture

$$Z_{N=4\,SYM}[J_i] = e^{-S_{IIB}[\Phi \to J_i]}$$

Very "interacting" quantity

complicated

classical calculation

$\lambda = R^4/\alpha'^2 \to \infty$

$g_s \to 0 \rightsquigarrow N_c \to \infty$

"Simple"

Wilson loop

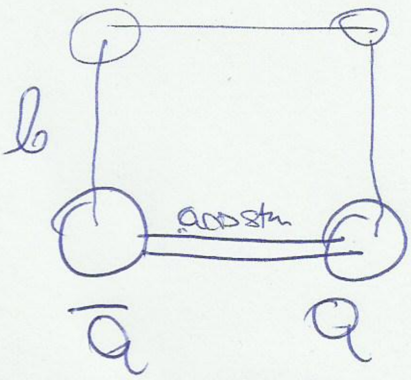
Alkofer Greensite
hep-ph 0610365

Greensite
What is confinement
hep-lat/0301023

Wilson 1974 → "Confinement of quarks"
read it

(invents Lattice field theory)

$$\mathcal{O} = \mathrm{Tr}\, e^{\oint_C A_\mu dx^\mu}$$



$$\langle \mathcal{O} \rangle \sim e^{-E_{q\bar q}}$$

Confinement? $\longrightarrow \lim_{L\to\infty} V_{q\bar q} \underset{L\to\infty}{\sim} \sigma L_{q\bar q}$

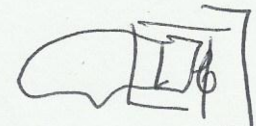
+ subleading

What is $V_{q\bar q}(L_{q\bar q})$?

Let us start one example (Maldacena 1998
Rey-Yee 1998)

Wilson loop

$$\langle W \rangle = \langle P\, e^{\oint_C A_\mu dx^\mu} \rangle \overset{\text{conjecture}}{\equiv} Z_{\text{string}}\Big[\quad \Big]$$



$$\sim e^{-S_{\text{string}}^{\text{with BC}}}$$

$$S_{\text{string}} = S_{NG} = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{\det g_{\alpha\beta}}$$

$$\boxed{g_{\alpha\beta} = G_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu.}$$

Classical Mechanics problem

Calcule en < any metric / AdS

(Sonnenschein 1999)

for "any" metric

Sonnenschein
1999 Lectures
Santiago Compostela

$$ds^2 = -g_{tt}\, dt^2 + g_{xx}\, dx^2 + g_{\rho\rho}\, d\rho^2 + \dots$$

$x = \sigma$

$t = \tau$

$\rho = \rho(\sigma)$

In AdS_5

$g_{tt} = g_{xx} = \rho^2/L^2$

$g_{\rho\rho} = L^2/\rho^2$

$$ds^2_{ind} = -g_{tt}\, d\tau^2 + g_{xx}\, d\sigma^2 + g_{\rho\rho}\, \rho'^2\, d\sigma^2$$

$$ds^2_{ind} = -g_{tt}\, d\tau^2 + (g_{xx} + g_{\rho\rho}\, \rho'^2)\, d\sigma^2$$

$f^2_{AdS_5} = \rho^4/L^4$

$g^2_{AdS_5} = 1$

$$-\det g_{ind} = g_{tt}\,(g_{xx} + g_{\rho\rho}\, \rho'^2) = f^2 + g^2 \rho'^2$$

case of AdS_5

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_0^T d\tau \int_0^{\ell} d\sigma \sqrt{f^2 + g^2 \rho'^2} = \int\int \sqrt{\rho^4/L^4 + \rho'^2}$$

Wilson loop on $AdS_5 \times S^5$

$$ds^2 = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2 dr^2}{r^2} + \dots \qquad f^2 = \frac{r^4}{L^4},\ g^2 = 1$$

$$x = \sigma \qquad t = \tau$$

$$r = r(\sigma)$$

$$ds_{ind}^2 = \frac{r^2}{L^2}(d\tau^2 + dx^2) + \frac{L^2}{r^2} r'^2 dx^2 \quad \rightsquigarrow \quad \begin{array}{c|cc} & \sigma & \tau \\ \hline \sigma & \frac{r^2}{L^2}\left(1+\frac{L^4}{r^4} r'^2\right) & 0 \\ \tau & 0 & -\frac{r^2}{L^2} \end{array}$$

$$-\det g_{ind} = \frac{r^2}{L^2}\frac{r^2}{L^2}\left(1 + \frac{L^4}{r^4} r'^2\right)$$

$$S_{NG} = \frac{T}{2\pi\alpha'} \int d\sigma \, \frac{r^2}{L^2} \sqrt{1 + \frac{L^4}{r^4} r'^2} = \int \sqrt{f^2 + g^2 r'^2}$$

$$P_r = \frac{T}{2\pi\alpha'} \frac{r^2}{L^2} \frac{1}{\sqrt{1+\frac{L^4}{r^4} r'^2}} \frac{L^4}{r^4} = \frac{T}{2\pi\alpha'} \frac{L^2}{r^2} \frac{r'}{\sqrt{1 + \frac{r^4}{L^4} r'^2}}$$

$$H = p_r \dot{r}' - L = \frac{T}{2\pi\alpha'}\left\{ \frac{L^2}{r^2} \frac{r'^2}{\sqrt{}} - \frac{r^2}{L^2}\sqrt{} \right\}$$

$$H = \frac{T}{2\pi\alpha'}\left\{ \frac{\cancel{\frac{L^2}{r^2} r'^2} - \frac{r^2}{L^2}\left(1 + \cancel{\frac{L^4}{r^4} r'^2}\right)}{\sqrt{1 + \frac{L^4}{r^4} r'^2}} \right\} = + \frac{T}{\cancel{2\pi\alpha}} \frac{r^2}{L^2\sqrt{1 + \frac{L^4}{r^4} r'^2}} = \frac{r_0^2}{L^2}\left(\cancel{\frac{+T}{2\pi\alpha}}\right)$$

$$r^4 = r_0^4 \left(1 + \frac{L^4}{r^4} r'^2\right) \Rightarrow \frac{r^4}{r_0^4} - 1 = \frac{L^4}{r^4} r'^2$$

$$r'^2 = \frac{r^4}{L^4}\left(-1 + \frac{r^4}{r_0^4}\right) \longrightarrow \frac{dr}{d\sigma} = \sqrt{(r^4 - r_0^4)\frac{r^4}{L^4}}$$

$$L(p) = 2\int d\sigma = 2\int \frac{L^2 \, dr}{r^2 \sqrt{r^4 - r_0^4}} \quad \checkmark$$

Energy

$$E_{QQ} = S_{QQ} = \int_{0} d\sigma \sqrt{\frac{r^4}{L^4} + r'^2}$$

use

$$r'^2 = (r^4 - r_0^4)\frac{r^4}{L^4} \qquad d\sigma = \frac{dr}{\sqrt{(r^4 - r_0^4)\frac{r^4}{L^4}}}$$

$$E_{QQ} = \int \frac{dr}{\sqrt{(r^4 - r_0^4)\frac{r^4}{L^4}}} \cancel{\sqrt{\frac{r^4}{L^4} + \frac{r^8}{L^4} - \frac{r^4 r_0^4}{L^4}}} \sqrt{\frac{r^4}{L^4} + (r^4 - r_0^4)\frac{r^4}{L^4}}$$

$$S = \frac{T}{2\pi\alpha'} \int d\sigma \, \frac{r^2}{L^2} \sqrt{1 + \frac{L^4}{r^4} r'^2}$$

$$p_r = \frac{\partial \mathcal{L}}{\partial r'} = \frac{T}{2\pi\alpha'} \frac{r^2}{L^2} \frac{1}{\sqrt{1 + \frac{L^4}{r^4} r'^2}} \frac{L^4}{r^4} r'$$

$$\boxed{p_r = \frac{T}{2\pi\alpha'} \frac{L^2}{r^2} \frac{r'}{\sqrt{1 + \frac{L^4}{r^4} r'^2}}}$$

$$H = p_r r' - \mathcal{L} = \frac{T}{2\pi\alpha'} \left\{ \frac{L^2}{r^2} \frac{r'^2}{\sqrt{1 + \frac{L^4}{r^4} r'^2}} - \frac{r^2}{L^2} \sqrt{1 + \frac{L^4}{r^4} r'^2} \right\}$$

$$= \frac{T}{2\pi\alpha'} \frac{L^2}{r^2} \left\{ \frac{r'^2}{\sqrt{1 + \frac{L^4}{r^4} r'^2}} - \frac{r^4}{L^4} \sqrt{1 + \frac{L^4}{r^4} r'^2} \right\}$$

$$= \frac{T}{2\pi\alpha'} \frac{L^2}{r^2} \frac{1}{\sqrt{1 + \frac{L^4}{r^4} r'^2}} \left\{ \cancel{r'^2} - \frac{r^4}{L^4} \left(1 + \cancel{\frac{L^4}{r^4} r'^2} \right) \right\}$$

$$= \frac{T}{2\pi\alpha'} \frac{r^2}{L^2 \sqrt{1 + \frac{L^4}{r^4} r'^2}} = \text{constant} = \frac{T}{2\pi\alpha'} \frac{r_0^2}{L^2}$$

$$\frac{r^2}{\cancel{L^2}}\sqrt{1+\frac{L^4}{r^4}r'^2} = \frac{r_0^2}{\cancel{L^2}}$$

$$r^4 = r_0^4\left(1+\frac{L^4}{r^4}r'^2\right)$$

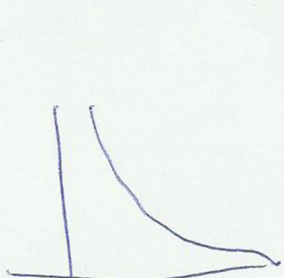
$$\frac{r^4}{r_0^4} = 1+\frac{L^4}{r^4}r'^2$$

$$\sqrt{\frac{r^4}{L^4}\left(\frac{r^4}{r_0^4}-1\right)} = \frac{dr}{d\sigma}$$

$$\frac{dr}{\sqrt{\frac{r^4}{L^4}\left(\frac{r^4}{r_0^4}-1\right)}} = d\sigma$$

$$\longrightarrow L_{q\bar{q}} = L^2 r_0^2 \int \frac{dr}{\sqrt{r^4(r^4-r_0^4)}} = (2\pi)^{3/2}\frac{\ell_{AdS}^2}{r_0\,\Gamma(1/4)^2}$$

$$L_{q\bar{q}}(r_0) \sim \frac{1}{r_0}$$



Energy

$$S = E = \int_0^{2\pi} d\sigma \sqrt{\frac{r^4}{L^4} + r'^2}$$

use $r'^2 \cong (r^4 - r_0^4)\,\frac{r^4}{\ell^4}$

$$d\sigma = \frac{dr}{\sqrt{\frac{r^4}{L^4}\left(\frac{r^4}{r_0^4}-1\right)}}$$

$$E = \int \cdots \quad = -\frac{(2\pi)^3\,\ell_{AdS}^2}{\Gamma(1/4)^4}\,\frac{1}{L_{QQ}}$$

$$E_{QQ} \sim \frac{C}{L_{QQ}} \qquad \boxed{C = \sim\sqrt{\lambda}}$$

Let us discuss the formalism of Wilson loops in a generic metric

$AdS_5 \times S^5$

$g_{tt} = -\frac{r^2}{L^2}, \; g_{xx} = \frac{r^2}{L^2}, \; g_{\rho\rho} = \frac{L^2}{r^2}$

$$ds^2 = -g_{tt}\, dt^2 + g_{xx}\, dx^2 + g_{\rho\rho}\, d\rho^2 + \dots$$

$$\left(g_{\alpha\beta} = G_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \qquad \alpha, \beta = \sigma, \tau \right)$$

we propose a configuration (as before)

$$\left.\begin{array}{l} x = \sigma \\ \rho = \rho(\sigma) \\ t = \tau \end{array}\right\} \quad ds^2_{ind} = -g_{tt}\, d\tau^2 + \left(g_{xx} + g_{\rho\rho}\, \rho'^2\right) d\sigma^2$$

$$g_{\alpha\beta} = \begin{array}{c|cc} & \tau & \sigma \\ \hline \tau & -g_{tt} & 0 \\ \sigma & 0 & \left(g_{xx} + g_{\rho\rho}\, \rho'^2\right) \end{array} \; ; \quad \underline{AdS} \begin{bmatrix} -\frac{r^2}{L^2} & 0 \\ 0 & \frac{r^2}{L^2}\left(1 + \frac{L^4 r'^2}{r^4}\right) \end{bmatrix}$$

$$\longrightarrow -\det g_{\alpha\beta} = g_{tt}\left(g_{xx} + g_{\rho\rho}\, \rho'^2\right) \rightsquigarrow \det g_{AdS} = -\frac{r^2}{L^2} \frac{r^2}{L^2}\left(1 + \frac{L^4 r'^2}{r^4}\right)$$

$$S_{NG} = \int \sqrt{g_{tt} g_{xx} + g_{tt} g_{\rho\rho}\, \rho'^2}\; \frac{d\sigma\, d\tau}{2\pi\alpha'} \rightsquigarrow \text{define} \quad \begin{array}{l} f^2 = g_{tt}\, g_{xx} \\ g^2 = g_{tt}\, g_{\rho\rho} \end{array}$$

$f^2 = \frac{r^4}{L^4}$

$g^2 = 1$

$S = \int d\sigma\, d\tau \sqrt{\frac{r^4}{L^4} + r'^2}$

$$S_{NG} = \frac{T}{2\pi\alpha'} \int d\sigma \cdot \sqrt{f^2_{(\rho)} + g^2_{(\rho)}\, \rho'^2} \longrightarrow$$

1-d system in classical mechanics $\longrightarrow$ conserved Energy with the Nambu-Goto Action

$$S = \frac{T}{2\pi\alpha'}\int d\sigma \sqrt{\frac{n^4}{\Sigma^4} + n'^2} \quad ; \quad p_n = \frac{n'}{\sqrt{\frac{n^4}{\Sigma^4}+n'^2}}$$

$$S_{NG} = \frac{T}{2\pi\alpha'}\int d\sigma \sqrt{f^2 + g^2 \rho'^2} \longrightarrow H = p_\rho \rho' - \mathcal{L}$$

$$H = \frac{n'^2}{\sqrt{\ }} - \sqrt{\ } = \frac{\frac{n^4}{\Sigma^4}}{\sqrt{\frac{n^4}{\Sigma^4}+n'^2}}$$

$$H = -\frac{f^2}{\sqrt{f^2+g^2\rho'^2}} \overset{\text{constant}}{=} -f(\rho_0) = -f_0$$

⇝ from here

$$\left[\frac{d\rho}{dx} = \frac{1}{f_0}\frac{f}{g}\sqrt{f^2 - f_0^2}\right]$$

⇝ then the separation between $Q\bar{Q}$ pair is

$$L_{Q\bar{Q}} = 2\int d\sigma = 2.\int_{\rho_0}^{\infty} d\rho \, \frac{f_0\, g}{f\sqrt{f^2 - f_0^2}}$$

$$\boxed{L_{Q\bar{Q}}^{(\rho_0)} = 2 f_0 \int_{\rho_0}^{\infty} d\rho \, \frac{g}{f\sqrt{f^2-f_0^2}}}$$

⇝ Let us now compute the Energy

$$E_{Q\bar{Q}} = \underbrace{\int d\sigma \sqrt{f^2 + g^2 \rho'^2}}_{\text{original } \frac{\text{NG Action}}{T}} - 2 \underbrace{\int_0^\infty g(\rho)\, d\rho}_{\text{mass of 2 straight strings}}$$

using as derived above $\left[\dfrac{d\rho\, f_0\, g}{f\sqrt{f^2 - f_0^2}} = d\sigma \right]$

$$E_{Q\bar{Q}} = 2 f(\rho_0) \cdot \int_{\rho_0}^\infty \frac{g/f}{\sqrt{f^2 - f_0^2}}\, d\rho - 2 \int_0^\infty g\, d\rho$$

or using the expression for $L_{Q\bar{Q}}(\rho_0)$

$$\boxed{E_{Q\bar{Q}}(\rho_0) = f(\rho_0)\, L_{Q\bar{Q}}(\rho_0) + 2 \int_{\rho_0}^\infty \frac{g}{f}\left[\sqrt{f^2 - f_0^2} - f\right] d\rho - 2 \int_0^{\rho_0} g\, d\rho}$$

Notice:
if $f(\rho_0) \neq 0$
$E \approx f(\rho_0)\, L_{Q\bar{Q}}$
→ confinement!