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Norwegian University of
Science and Technology

QCD phase diagram

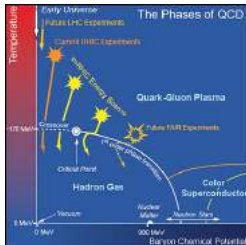
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Jens O. Andersen

1. Collaborators: Prabal Adhikari (St Olaf)
Patrick Kneschke (UiS)
2. References: Phys. Rev. D97 076005 (2018), Phys. Rev. D95, 036017 (2017)
Phys.Rev. D98 074016 (2018), Phys.Rev. D97 076005 (2018)
arXiv:1807.08951

Why QCD in extreme conditions?

1. Early universe
2. Heavy-ion collisions
3. Compact stars
4. Phase diagram of QCD



1. QCD is the fundamental theory of strong interactions

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi}_f [i\gamma^\mu D_\mu - m]\psi_f . \quad (1)$$

- a) Symmetries of massless QCD

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_A \times SU(N_c) \quad (2)$$

- b) Symmetries spontaneously broken in the vacuum

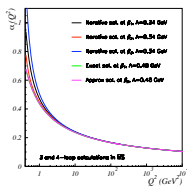
$$SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V \quad (3)$$

- c) Pions are the (pseudo)-Goldstone bosons

- d) $U(1)_A$ broken by instantons. η' not a GB

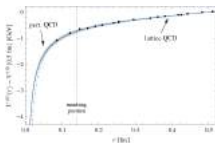
e) Asymptotic freedom

$$\alpha_s(Q) = \frac{12\pi}{(33 - 22N_f) \log(Q^2/\Lambda^2)} \quad (4)$$



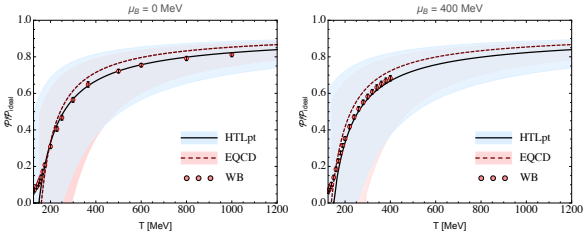
f) Confinement

$$V(r) = -\frac{4\alpha_s}{r} + br. \quad (5)$$



1. Only a few exact results of the phase diagram are known
 - a) Quark-gluon plasma at asymptotically large temperatures
 - b) Color-flavor locked phase at asymptotically large densities
 - c) Lattice calculation for finite T but not finite μ_B
 - d) Lattice calculations for finite μ_I and finite B (but not the combination)
 - e) Model calculations for small and baryon intermediate densities
 - f) Order of phase transitions? Cross-overs and critical point?

1. Pressure



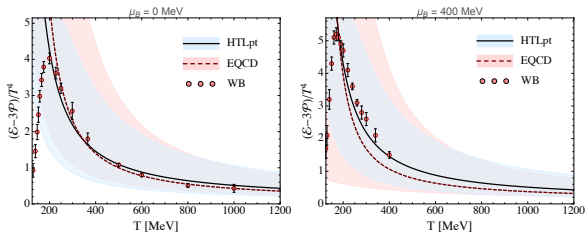
2. Expansion of pressure

$$\frac{\mathcal{P}}{T^4} = \frac{\mathcal{P}_0}{T^4} + \sum_{ijk} \frac{1}{i!j!k!} \chi_{ijk} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

Quark susceptibilities

$$\chi_{ijk\dots} = \left. \frac{\partial^{i+j+k+\dots} \mathcal{P}(T, \mu)}{\partial \mu_u^i \partial \mu_d^j \partial \mu_s^k} \right|_{\mu_q=0}.$$

3. Trace anomaly



$$\frac{\varepsilon - 3\mathcal{P}}{T^4} \quad (6)$$

Theories and models for QCD

1. Lattice gauge theories. Discretizing space-time on a computer.

a) In principle exact, but...

b) Sign problem for finite μ_B

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu e^{-\int d^3x \int_0^\beta \bar{\psi} [\not{D} + m_f] \psi} e^{-S_g} \\ &= \int \mathcal{D}A_\mu e^{-S_g} \det[\not{D} + m_f] . \end{aligned} \quad (7)$$

c) No sign problem for finite isospin or magnetic field

$$\det[\not{D} + m_f] = \det \left[X X^\dagger + m_f^2 \right] . \quad (8)$$

2. Low-energy effective theory for QCD. Includes pions (and baryons).

a) Systematic approach based on symmetries and degrees of freedom

b) Pions are derivatively coupled.

3. Chiral models: Nambu-Jona-Lasinio model and Quark-meson model
 - a) (non)-renormalizable
 - b) Not a systematic approach (?)
4. Models used in nuclear physics such as σ - ω models (Dexheimer's lectures)

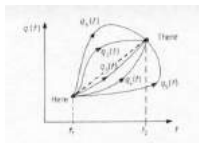
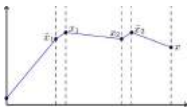
Imaginary-time formalism

1. Transition amplitude

$$\langle \phi_b | e^{-iH(t_b - t_a)} | \phi_a \rangle = \int \mathcal{D}\phi e^{iS}, \quad (9)$$

$$S = \int_{t_a}^{t_b} dt \int d^3x \mathcal{L} \quad (10)$$

2. Paths in quantum mechanics



3. Partition function

$$\begin{aligned}
 Z &= \text{Tr} e^{-\beta H} \\
 &= \sum_a \langle \phi_a | e^{-\beta H} | \phi_a \rangle
 \end{aligned} \tag{11}$$

4. Imaginary time, $\tau = it$. Path-integral representation

$$Z = \int_{\text{periodic}} \mathcal{D}\phi e^{-S_E} , \tag{12}$$

$$S_E = \int_0^\beta d\tau \int d^3x \mathcal{L}_E . \tag{13}$$

5. Fourier expansion

$$\phi(t, \mathbf{x}) = \frac{1}{\sqrt{\beta V}} \sum_n \sum_{\mathbf{p}} \phi_{n,\mathbf{p}} e^{-i(\omega_n t + \mathbf{p} \cdot \mathbf{x})} . \tag{14}$$

6. Partition function free theory

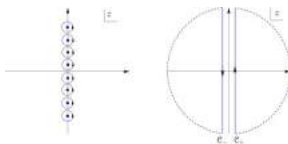
$$Z = \prod_{n,\mathbf{p}} \int \mathcal{D}\phi_{n,\mathbf{p}} \exp \left[-\frac{1}{2} \phi_{n,\mathbf{p}}^* (\omega_n^2 + \omega^2) \phi_{n,\mathbf{p}} \right] , \tag{15}$$

7. Gaussian integrals

$$Z = \sum_{n, \mathbf{p}} \left[\beta^2 (\omega_n^2 + \omega^2) \right]^{-\frac{1}{2}}, \quad (16)$$

$$\log Z = -\frac{1}{2} \sum_{n, \mathbf{p}} \log \left[\beta^2 (\omega_n^2 + \omega^2) \right]. \quad (17)$$

8. Matsubara sums using complex integration (exercise!)



$$\begin{aligned} \frac{1}{V} \sum_P \int_P \log [P^2 + m^2] &= \int_p \frac{d^3 p}{(2\pi)^3} \sqrt{p^2 + m^2} \\ &+ 2T \int_p \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-\beta \sqrt{p^2 + m^2}} \right] \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{1}{V} \sum_P \int_P \log [P^2 + m^2] &= - \int_p \frac{d^3 p}{(2\pi)^3} \sqrt{p^2 + m^2} \\ &- 2T \int_p \frac{d^3 p}{(2\pi)^3} \log \left[1 + e^{-\beta \sqrt{p^2 + m^2}} \right] \end{aligned} \quad (19)$$

A simple model for Bose condensation

1. Lagrangian

$$\mathcal{L} = -(\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) - m^2 \Phi^\dagger \Phi - h(\Phi + \Phi^\dagger) - \frac{\lambda}{24} (\Phi^\dagger \Phi)^2$$

2. Conserved charge (calculate it!). Adding a chemical potential

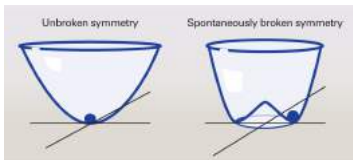
$$\partial_\mu \rightarrow D_\mu = \partial_\mu - \delta_\mu^0 \mu, \quad (20)$$

3. Assuming a condensate, $\langle \phi_1 \rangle = \rho_0 \neq 0$

$$V = \frac{1}{2} (m^2 - \mu^2) \rho_0^2 + \frac{\lambda}{24} \rho_0^4. \quad (21)$$

4. Symmetry breaking if $\mu^2 > m^2$

$$\rho_0 = \sqrt{\frac{6(\mu^2 - m^2)}{\lambda}}. \quad (22)$$



Classical potential for $m^2 > \mu^2$ (left) and $m^2 < \mu^2$ (right).

5. Inverse propagator

$$D_0^{-1}(k) = \begin{pmatrix} k^2 - m_1^2 & -2ik_0\mu \\ 2ik_0\mu & k^2 - m_2^2 \end{pmatrix}. \quad (23)$$

6. Dispersion relation

$$E_{\pm}(k) = k^2 + 3\mu^2 - m^2 \pm \sqrt{(3\mu^2 - m^2)^2 + 4p^2\mu^2} \quad (24)$$

7. Goldstone boson

$$E_-(k) = \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} k. \quad (25)$$

Linear sigma model

1. Lagrangian

$$\begin{aligned}
 \mathcal{L} &= -(\partial_\mu \phi_i)(\partial_\mu \phi_i) - m^2 \Phi_i \phi_i - h \phi_1 - \frac{\lambda}{24} (\phi_i \phi_i)^2 \\
 &= -\text{Tr} \left[(\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) \right] - m^2 \text{Tr} \left[\Phi^\dagger \Phi \right] - h \text{Tr} [\Phi + \Phi^\dagger] \\
 &\quad - \frac{\lambda}{24} \text{Tr} \left[(\Phi^\dagger \Phi) \right]^2
 \end{aligned} \tag{26}$$

2. $O(4)$ -symmetry. Describes three pion and the sigma.

3. Vacuum expectation value

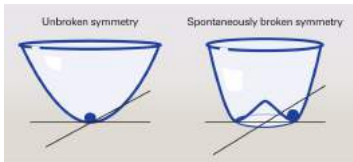
$$\Phi_1 = \phi_0 + \tilde{\Phi}_1 . \tag{27}$$

4. Classical potential

$$V = \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda}{24} \phi_0^4 - h \phi_0 \tag{28}$$

5. Spontaneous symmetry breaking if $m^2 < 0$

$$\phi_0 = \sqrt{-\frac{6m^2}{\lambda}} \equiv f_\pi . \tag{29}$$



Classical potential for $m^2 > 0$ (left) and $m^2 < 0$ (right).

6. Goldstone bosons are angular excitations

7. Adding a chemical potential

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - \delta_\mu^0 \mu_I \quad (30)$$

$$V = \frac{1}{2} m^2 \phi_0^2 + \frac{1}{2} (m^2 - \mu_I^2) \rho_0^2 - h \phi_0 + \frac{\lambda}{24} (\phi_0^2 + \rho_0^2)^2 \quad (31)$$

8. Parameter fixing

$$m^2 = -\frac{1}{2} (m_\sigma^2 - m_\pi^2) , \quad (32)$$

$$\lambda = \frac{(m_\sigma^2 - m_\pi^2)}{3 f_\pi^2} , \quad (33)$$

$$h = m_\pi^2 f_\pi . \quad (34)$$

9. Condensate in the vacuum phase

$$\phi_0 = f_\pi , \quad (35)$$

$$\rho_0 = 0 . \quad (36)$$

10. Condensates in pion-condensed phase

$$\phi_0 = \frac{h}{\mu_I^2} = \frac{m_\pi^2 f_\pi}{\mu_I^2} , \quad (37)$$

$$\rho_0^2 = \frac{2}{\lambda} (\mu_I^2 - m^2) - \frac{h^2}{\mu_I^4} . \quad (38)$$

11. Pion condensate

$$\rho_0^2 = \frac{(2\mu_I^2 + m_\sigma^2 - 3m_\pi^2)f_\pi^2}{m_\sigma^2 - m_\pi^2} - \frac{m_\pi^4 f_\pi^2}{\mu_I^4}. \quad (39)$$

12. Pressure

$$P = \frac{1}{8} f_\pi^2 \frac{4m_\sigma^2(\mu_I^2 - 2m_\pi^2) + (2\mu_I^2 - 3m_\pi^2)^2 + 3m_\pi^4}{m_\sigma^2 - m_\pi^2} + \frac{1}{2} \frac{m_\pi^4 f_\pi^2}{\mu_I^2}.$$

13. Energy density

$$E = \frac{1}{2} f_\pi^2 \frac{m_\sigma^2(\mu_I^2 + 2m_\pi^2) + 3\mu_I^2(\mu_I^2 - m_\pi^2) - 3m_\pi^4}{m_\sigma^2 - m_\pi^2} - \frac{3}{2} \frac{m_\pi^4 f_\pi^2}{\mu_I^2}.$$

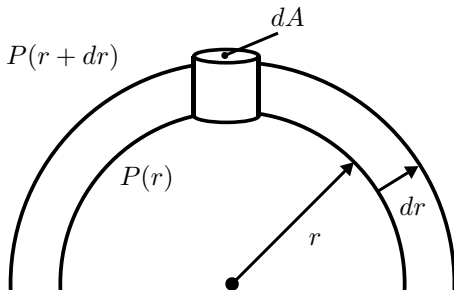
14. Chiral perturbation theory ($m_\sigma \rightarrow \infty$)

$$\rho_0^{\text{CHPT}} = f_\pi \sqrt{1 - \frac{m_\pi^4}{\mu_I^4}}, \quad (40)$$

$$P^{\text{CHPT}} = \frac{1}{2} f_\pi^2 \mu_I^2 \left(1 - \frac{m_\pi^2}{\mu_I^2}\right)^2. \quad (41)$$

Pion stars

1. Structure equations (TOV equations)

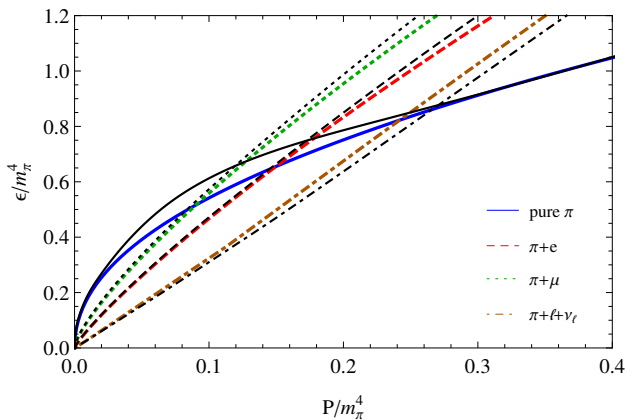


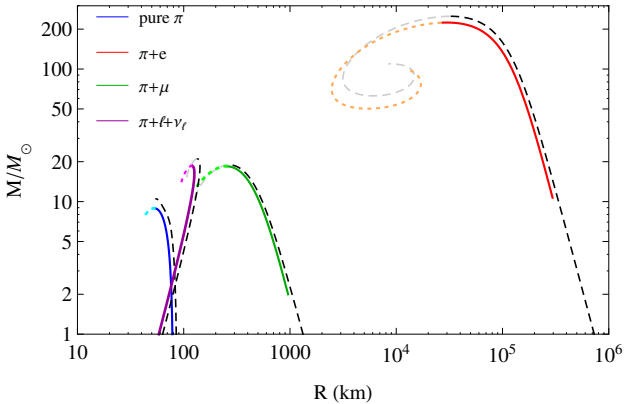
$$\frac{dM}{dr} = \frac{4\pi r^2 E(r)}{c^2}, \quad (42)$$

$$\begin{aligned} \frac{dP(r)}{dr} = & -\frac{GE(r)M(r)}{c^2 r^2} \left[1 + \frac{P(r)}{E(r)} \right] \left[1 + \frac{4\pi r^3 P(r)}{M(r)c^2} \right] \\ & \times \left[1 - \frac{2GM(r)}{c^2 r} \right]^{-1}. \end{aligned} \quad (43)$$

2. We need the equation of state

3. Equation of state





Mass-radius relations for boson stars

4. Inverse propagators

$$D_0^{-1}(k) = \begin{pmatrix} k^2 - m_1^2 & \frac{\lambda}{3} \phi_0 \rho_0 & 0 & 0 \\ \frac{\lambda}{3} \phi_0 \rho_0 & k^2 + m_2^2 & -2i\mu_I p_0 & 0 \\ 0 & 2i\mu_I p_0 & k^2 - m_3^2 & 0 \\ 0 & 0 & 0 & k^2 - m_4^2 \end{pmatrix} \quad (44)$$

5. Dispersion relations



Quark content of a π^+

$$\pi^+ \rightarrow \mu^+ + \nu_\mu, \quad (45)$$

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu. \quad (46)$$



Pion decay

6. Local charge neutrality

$$n_Q = 0. \quad (47)$$

7. Chemical equilibrium

$$\mu_I = \mu_Q , \quad (48)$$

$$\mu_e = -\mu_Q + \mu_{L_e} , \quad (49)$$

$$\mu_{\nu_e} = \mu_{L_e} , \quad (50)$$

$$\mu_\mu = -\mu_Q + \mu_{L_\mu} , \quad (51)$$

$$\mu_{\nu_\mu} = \mu_{L_\mu} . \quad (52)$$

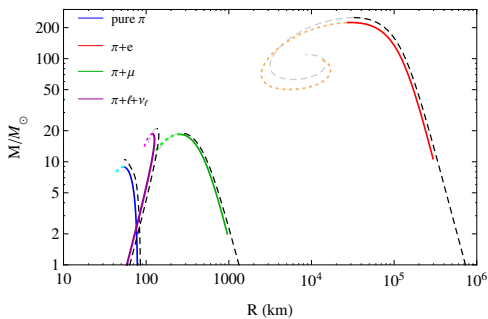
8. Pressure of free leptons

$$\begin{aligned} P_l &= 2 \int \frac{d^3 q}{(2\pi)^3} \left(\mu_l - \sqrt{q^2 + m_l^2} \right) \Theta \left(\mu_l - \sqrt{q^2 + m_l^2} \right) \\ &= \frac{4}{3(4\pi)^2} \left\{ \mu_l \sqrt{\mu_l^2 - m_l^2} (\mu_l^2 - \frac{5}{2} m_l^2) \right. \\ &\quad \left. + \frac{3}{2} m_l^4 \log \frac{\mu_l + \sqrt{\mu_l^2 - m_l^2}}{m_l} \right\} \Theta(\mu_l - m_l) . \end{aligned} \quad (53)$$

9. Number density

$$n_l(\mu_l) = \frac{dP_l}{d\mu_l} = \frac{16}{3(4\pi)^2} \left(\mu_l^2 - m_l^2 \right)^{\frac{3}{2}} \Theta(\mu_l - m_l) \quad (54)$$

1. Mass-radius relation



2. Total mass M , radius R , central pressure P_c , central isospin chemical potential $\mu_{I,c}$, surface pressure P_s , and total neutrino density $n_\nu = n_{\nu_e} + n_{\nu_\mu}$ at the surface for the largest stable stars in appropriate units

System	π	$\pi + \mu$	$\pi + e$	$\pi + l + \nu_l$
$M/M_\odot$	8.91	18.45	223.86	18.65
R [km]	53.06	252.08	2.724×10^4	118.9
P_c [MeV/fm ³]	20.6	0.23	4.56×10^{-7}	3.75
$\mu_{I,c}$ [MeV]	253.88	142.0928	140.0000851	140.905
P_s [MeV/fm ³]	0	0	0	0.43
n_ν [1/fm ³]	0	0	0	1.22×10^{-3}

Quark-meson model

1. Add fermions to the linear sigma model. Possible to address questions involving nonzero baryon density
2. Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{meson}} = & \frac{1}{2} \left[(\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2 \right] + \frac{1}{2} m^2 \left[\sigma^2 + \boldsymbol{\pi}^2 \right] \\ & + \frac{\lambda}{24} \left[\sigma^2 + \boldsymbol{\pi}^2 \right]^2 - h \sigma , \end{aligned} \quad (55)$$

$$\mathcal{L}_{\text{quark}} = \bar{\psi} \left[\gamma_\mu \partial_\mu - \mu \gamma_4 \right] \psi \quad (56)$$

$$\mathcal{L}_{\text{Yukawa}} = \bar{\psi} \left[g(\sigma - i \gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) \right] \psi . \quad (57)$$

3. Symmetries (cf QCD and instantons)

$$SU(2)_L \times SU(2)_R \times U(1)_B \times SU(N_c) . \quad (58)$$

4. If $\langle \sigma \rangle = \phi_0 \neq 0$

$$SU(2)_V \times U(1)_B \times SU(N_c) . \quad (59)$$

5. Adding an isospin chemical potential further breaks the symmetries.

$$SU(2)_L \times SU(2)_R \rightarrow U(1) \times U(1) , \quad (60)$$

$$SU(2)_V \rightarrow U(1) . \quad (61)$$

6. Can now address questions of chiral symmetry restoration and pion condensation

7. Approximations

a) Loop expansion (problematic without resummation)

b) $1/N_c$ -expansion (effective action)

c) Functional renormalization group

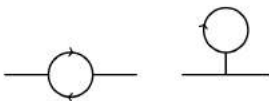
8. Parameter fixing

- a) Tree-level relations no longer hold
- b) Must match the parameters to physical quantities at the same level of accuracy as the effective potential!
- c) Use on-shell and $\overline{\text{MS}}$ schemes
- d) Self energies and renormalization conditions

$$p^2 - m_{\sigma,\pi}^2 - i\Sigma_{\sigma,\pi}(p^2) + \text{counterterms} \quad (62)$$

$$\Sigma^{\text{os}}(p^2 = m_{\sigma,\pi}^2) + \text{counterterms} = 0, \quad (63)$$

$$\left. \frac{\partial}{\partial p^2} \Sigma_{\sigma,\pi}(p^2) \right|_{p^2 = m_{\sigma,\pi}^2} + \text{counterterms} = 0. \quad (64)$$



Self-energy of the σ particle

e) Tadpoles and renormalization condition

$$\delta\Gamma^{(1)} = 0. \quad (65)$$



Tadpole diagram for the sigma particle and the counterterm

f) Counterterms determined by renormalization conditions

$$\delta m_{os}^2 = \delta m_{div}^2 + \dots \quad (66)$$

$$\delta \lambda_{os} = \delta \lambda_{div} + \dots \quad (67)$$

g) Relations among renormalized parameters

$$m_{ms}^2 = m_{os}^2 + \delta m_{os}^2 - \delta m_{ms}^2, \quad (68)$$

$$\lambda_{ms} = \dots \quad (69)$$

9. Renormalized parameters in terms of physical masses etc

$$m_{ms}^2 = -\frac{1}{2} m_{\sigma}^2 \left\{ 1 + \frac{4m_q^2 N_c}{(4\pi)^2 f_{\pi}^2} \left[\log \left(\frac{\Lambda^2}{m_q^2} \right) + \frac{4m_q^2}{m_{\sigma}^2} + \left(1 - \frac{4m_q^2}{m_{\sigma}^2} \right) F(m_{\sigma}^2) \right] \right\}, \quad (70)$$

$$\lambda_{ms} = \frac{3m_{\sigma}^2}{f_{\pi}^2} \left\{ 1 + \frac{4m_q^2 N_c}{(4\pi)^2 f_{\pi}^2} \left[\left(2 - \frac{4m_q^2}{m_{\sigma}^2} \right) \log \left(\frac{\Lambda^2}{m_q^2} \right) + \left(1 - \frac{4m_q^2}{m_{\sigma}^2} \right) F(m_{\sigma}^2) \right] \right\}, \quad (71)$$

1. Vacuum contribution to the one-loop effective potential

$$V_1 = -4N_c \int_p \sqrt{p^2 + \Delta^2} = \frac{2N_c \Delta^4}{(4\pi)^2} \left(\frac{\Lambda^2}{\Delta^2} \right)^\epsilon \left[\frac{1}{\epsilon} + \frac{3}{2} \right]. \quad (72)$$

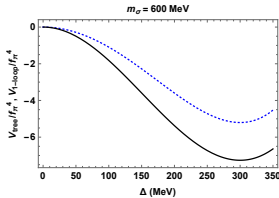
2. Thermodynamic potential

$$V_1 = \frac{1}{2} m_{\text{ms}}^2(\Lambda) \frac{\Delta^2}{g_{\text{ms}}^2(\Lambda)} + \frac{\lambda_{\text{ms}}(\Lambda)}{24} \frac{\Delta^4}{g_{\text{ms}}^4(\Lambda)} + \frac{2N_c \Delta^4}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\Delta^2} + \frac{3}{2} \right] \\ - 4N_c T \int_p \left\{ \log \left[1 + e^{-\beta(E-\mu)} \right] + \log \left[1 + e^{-\beta(E+\mu)} \right] \right\},$$

Thermodynamics of the QM model

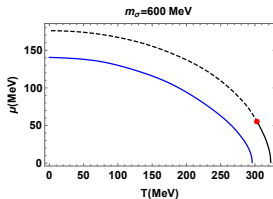
$$\begin{aligned}
 V_1 = & -\frac{1}{4} m_\sigma^2 f_\pi^2 \left\{ 1 + \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\left(1 - \frac{4m_q^2}{m_\sigma^2} \right) F(m_\sigma^2) + \frac{4m_q^2}{m_\sigma^2} \right] \right\} \frac{\Delta^2}{m_q^2} \\
 & + \frac{1}{8} m_\sigma^2 f_\pi^2 \left\{ 1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\frac{4m_q^2}{m_\sigma^2} \left(\log \left(\frac{\Delta^2}{m_q^2} \right) - \frac{3}{2} \right) \right. \right. \\
 & \left. \left. - \left(1 - \frac{4m_q^2}{m_\sigma^2} \right) F(m_\sigma^2) \right] \right\} \frac{\Delta^4}{m_q^4} \\
 & - 4N_c T \int_p \left\{ \log \left[1 + e^{-\beta(E-\mu)} \right] + \log \left[1 + e^{-\beta(E+\mu)} \right] \right\} \quad (73)
 \end{aligned}$$

3. Results



Vacuum effective potential normalized to f_π^4 as a function of Δ for $m_\sigma = 600$ MeV.

4. No-sea approximation ignores vacuum energy and renormalization



The phase diagram in the μ - T plane for $m_\sigma = 600$ MeV. A dashed line indicates a second-order transition, while a solid indicates a first-order transition. The red dot shows the tricritical point. The blue solid line is phase boundary in the no-sea approximation.

Pion condensation revisited

1. Vacuum energy

$$V_1 = -N_c \int_p \left(E_u + E_d + E_{\bar{u}} + E_{\bar{d}} \right) , \quad (74)$$

$$E_u = E(-\mu_I) , \quad E_d = E(\mu_I) , \quad (75)$$

$$E_{\bar{u}} = E(\mu_I) , \quad E_{\bar{d}} = E(-\mu_I) , \quad (76)$$

$$E(\mu_I) = \left[\left(\sqrt{p^2 + \Delta^2} + \mu_I \right)^2 + \rho^2 \right]^{\frac{1}{2}} . \quad (77)$$

2. Isolate UV divergences

$$\begin{aligned} V_{\text{div}} &= -4N_c \int_p \left[\sqrt{p^2 + \Delta^2 + \rho^2} + \frac{\mu_I^2 \rho^2}{2(p^2 + \Delta^2 + \rho^2)^{\frac{3}{2}}} \right] \\ &= \frac{4N_c}{(4\pi)^2} \left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2 + \rho^2} \right)^\epsilon \left[\left(\Delta^2 + \rho^2 \right)^2 \Gamma(-2 + \epsilon) \right. \\ &\quad \left. - 2\mu_I^2 \rho^2 \Gamma(\epsilon) \right] . \end{aligned} \quad (78)$$

The remainder V_{fin} is finite and reads

$$V_{\text{fin}} = V_1 - V_{\text{div}} . \quad (79)$$

3. Renormalized potential at $T = 0$

$$\begin{aligned}
 V_1 = & \frac{3}{4} m_\pi^2 f_\pi^2 \left\{ 1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} m_\pi^2 F'(m_\pi^2) \right\} \frac{\Delta^2 + \rho^2}{m_q^2} \\
 & - \frac{1}{4} m_\sigma^2 f_\pi^2 \left\{ 1 + \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\left(1 - \frac{4m_q^2}{m_\sigma^2} \right) F(m_\sigma^2) + \frac{4m_q^2}{m_\sigma^2} - F(m_\pi^2) - \right. \right. \\
 & \left. \left. m_\pi^2 F'(m_\pi^2) \right] \right\} \frac{\Delta^2 + \rho^2}{m_q^2} \\
 & - 2\mu_I^2 f_\pi^2 \left\{ 1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\log \frac{\Delta^2 + \rho^2}{m_q^2} + F(m_\pi^2) + m_\pi^2 F'(m_\pi^2) \right] \right\} \frac{\rho^2}{m_q^2} \\
 & + \frac{1}{8} m_\sigma^2 f_\pi^2 \left\{ 1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\frac{4m_q^2}{m_\sigma^2} \left(\log \frac{\Delta^2 + \rho^2}{m_q^2} - \frac{3}{2} \right) \right. \right. \\
 & \left. \left. - \left(1 - \frac{4m_q^2}{m_\sigma^2} \right) F(m_\sigma^2) + F(m_\pi^2) + m_\pi^2 F'(m_\pi^2) \right] \right\} \frac{(\Delta^2 + \rho^2)^2}{m_q^4} \\
 & - \frac{1}{8} m_\pi^2 f_\pi^2 \left[1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} m_\pi^2 F'(m_\pi^2) \right] \frac{(\Delta^2 + \rho^2)^2}{m_q^4} \\
 & - m_\pi^2 f_\pi^2 \left[1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} m_\pi^2 F'(m_\pi^2) \right] \frac{\Delta}{m_q} + V_{\text{fin}} . \tag{80}
 \end{aligned}$$

1. Wilson line

$$L(\mathbf{x}) = \mathcal{P} \exp \left[i \int_0^\beta d\tau A_4(\mathbf{x}, \tau) \right], \quad (81)$$

2. Polyakov loop

$$\Phi = \text{Tr} L(\mathbf{x}) . \quad (82)$$

3. Transforms under center transformations

$$L(\mathbf{x}) \rightarrow e^{2\pi i n / N_c} L(\mathbf{x}) . \quad (83)$$

4. Order parameter for confinement

$$\langle \Phi \rangle \simeq 0 \quad \text{confinement at low } T \quad (84)$$

$$\langle \Phi \rangle \simeq 1 \quad \text{deconfinement at high } T . \quad (85)$$

5. Background field A_4 in the Polyakov gauge

$$A_4 = t_3 A_4^3 + t_8 A_4^8, \quad (86)$$

6. Parametrization

$$L = \begin{pmatrix} e^{i(\phi_1 + \phi_2)} & 0 & 0 \\ 0 & e^{i(-\phi_1 + \phi_2)} & 0 \\ 0 & 0 & e^{-2i\phi_2} \end{pmatrix}, \quad (87)$$

7. Finite-temperature contribution

$$\begin{aligned} V_T &= -2T \int \frac{d^3 p}{(2\pi)^3} \left\{ \text{Tr} \log \left[1 + 3(\Phi + \bar{\Phi} e^{-\beta E u}) e^{-\beta E u} \right. \right. \\ &\quad \left. \left. + e^{-3\beta E u} \right] + \text{Tr} \log \left[1 + 3(\bar{\Phi} + \Phi e^{-\beta E \bar{u}}) e^{-\beta E \bar{u}} + e^{-3\beta E \bar{u}} \right] \right. \\ &\quad \left. + \text{Tr} \log \left[1 + 3(\Phi + \bar{\Phi} e^{-\beta E d}) e^{-\beta E d} + e^{-3\beta E d} \right] \right. \\ &= \left. + \text{Tr} \log \left[1 + 3(\bar{\Phi} + \Phi e^{-\beta E \bar{d}}) e^{-\beta E \bar{d}} + e^{-3\beta E \bar{d}} \right] \right\}. \quad (88) \end{aligned}$$

8. Distribution functions

$$n_F = \frac{1 + 2\bar{\Phi} e^{\beta E} + \Phi e^{2\beta E}}{1 + 3\bar{\Phi} e^{\beta E} + 3\Phi e^{2\beta E} + 3e^{\beta E}} \quad (89)$$

9. Pure-gluon potential

$$\frac{\mathcal{U}}{T^4} = -\frac{1}{2}a\Phi\bar{\Phi} + b \log \left[1 - 6\Phi\bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi\bar{\Phi})^2 \right], \quad (90)$$

with

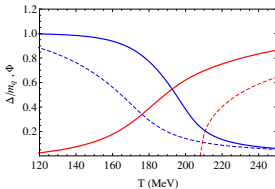
$$a = 3.51 - 2.47 \left(\frac{T_0}{T} \right) + 15.2 \left(\frac{T_0}{T} \right)^2, \quad (91)$$

$$b = -1.75 \left(\frac{T_0}{T} \right)^3. \quad (92)$$

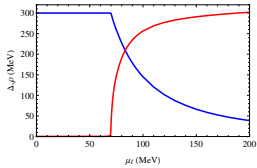
The temperature T_0 is defined by

$$T_0(N_f, \mu_I) = T_\tau e^{-1/(\alpha_0 b(\mu_I))}, \quad (93)$$

10. Condensates

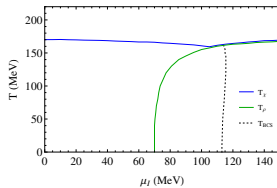


Normalized chiral condensate $\frac{\Delta}{m_q}$ and Polyakov-loop Φ as functions of the temperature T for $\mu_B = \mu_I = 0$



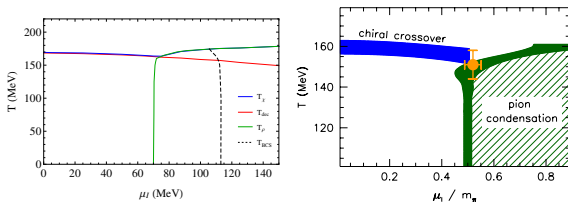
Chiral (red line) and pion condensates (blue line) Δ and ρ as functions of the isospin chemical potential μ_I at $T = 0$.

11. Phase diagram



Phase diagram in the μ_I - T plane for $\mu_B = 0$ without Polyakov loop.

12. Phase diagram - comparison with lattice results of Endrodi et al

Phase diagram in the μ_I - T plane for $\mu_B = 0$ with Polyakov loop.

Inhomogeneous condensate

1. Is it possible that a pion or chiral condensate is inhomogeneous?
2. Different ansätze for the condensate
3. Competition between condensates
4. Renormalization issues and possible artefacts of the regulator
5. Finite-temperature effects?

1. Chiral density wave

$$\sigma = \phi_0 \cos(qz) , \quad \pi_3 = \phi_0 \sin(qz) \quad (94)$$

$$\pi_1 = \pi_0 , \quad \pi_2 = 0 , \quad (95)$$

2. Tree-level potential with $\Delta = g\phi_0$ and $\rho = g\pi_0$

$$\begin{aligned} V_0 = & \frac{1}{2} f_\pi^2 q^2 \frac{\Delta^2}{m_q^2} - \frac{1}{4} f_\pi^2 (m_\sigma^2 - 3m_\pi^2) \frac{\Delta^2 + \rho^2}{m_q^2} - 2\mu_I^2 f_\pi^2 \frac{\rho^2}{m_q^2} \\ & + \frac{1}{8} f_\pi^2 (m_\sigma^2 - m_\pi^2) \frac{(\Delta^2 + \rho^2)^2}{m_q^4} \\ & - m_\pi^2 f_\pi^2 \frac{\Delta}{m_q} \cos(qz) \delta_{q,0} . \end{aligned} \quad (96)$$

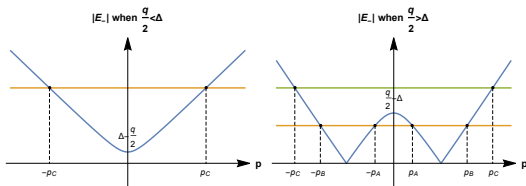
3. Dispersion relations

$$E_u^\pm = E(\pm q, -\mu_I) , \quad E_d^\pm = E(\pm q, \mu_I) , \quad (97)$$

$$E_{\bar{u}}^\pm = E(\pm q, \mu_I) , \quad E_{\bar{d}}^\pm = E(\pm q, -\mu_I) , \quad (98)$$

$$E(q, \mu_I) = \left[\left(\sqrt{p_\perp^2 + \left(\sqrt{p_\parallel^2 + \Delta^2 + \frac{q}{2}} \right)^2} + \mu_I \right)^2 + \rho^2 \right]^{\frac{1}{2}} \quad (99)$$

4. Dispersion relations



5. One-loop contribution

$$V_1 = -\frac{1}{2} N_c \int_p \left(E_u^\pm + E_d^\pm + E_{\bar{u}}^\pm + E_{\bar{d}}^\pm \right), \quad (100)$$

6. Isolating the divergences

$$\begin{aligned}
 V_{\text{div}} &= -4N_c \int_p \left[\sqrt{p^2 + \Delta^2 + \rho^2} + \frac{\mu_I^2 \rho^2}{2(p^2 + \Delta^2 + \rho^2)^{\frac{3}{2}}} \right. \\
 &\quad + \frac{3q^2 \mu_I^2 \rho^2 (4\Delta^2 + 4p_{\parallel}^2 - p_{\perp}^2 - \rho^2)}{16(p^2 + \Delta^2 + \rho^2)^{\frac{7}{2}}} \\
 &\quad \left. + \frac{q^2 (p_{\perp}^2 + \rho^2)}{8(p^2 + \Delta^2 + \rho^2)^{\frac{3}{2}}} + \frac{q^4 (p_{\perp}^2 + \rho^2) (4\Delta^2 + 4p_{\parallel}^2 - p_{\perp}^2 - \rho^2)}{128(p^2 + \Delta^2 + \rho^2)^{\frac{7}{2}}} \right] \\
 &= \frac{2N_c}{(4\pi)^2} \left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2 + \rho^2} \right)^{\epsilon} \left\{ 2 (\Delta^2 + \rho^2)^2 \Gamma(-2 + \epsilon) + q^2 \Delta^2 \Gamma(\epsilon) \right. \\
 &\quad - 4\mu_I^2 \rho^2 \Gamma(\epsilon) - 2q^2 \mu_I^2 \frac{\Delta^2 \rho^2}{(\Delta^2 + \rho^2)^2} \Gamma(2 + \epsilon) \\
 &\quad \left. - \frac{q^4}{12} \frac{\Delta^2}{(\Delta^2 + \rho^2)^2} \left[(1 - \epsilon) \Delta^2 + 2\rho^2 \right] \Gamma(1 + \epsilon) \right\} . \quad (101)
 \end{aligned}$$

The remainder V_{fin} is finite and reads

$$V_{\text{fin}} = V_1 - V_{\text{div}} . \quad (102)$$

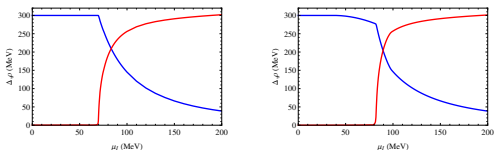
7.

8. Effective potential

$$\begin{aligned}
V_1 &= \frac{1}{2} f_\pi^2 q^2 \left\{ 1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} \left[\log \frac{\Delta^2 + \rho^2}{m_q^2} + F(m_\pi^2) + m_\pi^2 F'(m_\pi^2) \right] \right\} \\
&\quad - m_\pi^2 f_\pi^2 \left[1 - \frac{4m_q^2 N_c}{(4\pi)^2 f_\pi^2} m_\pi^2 F'(m_\pi^2) \right] \frac{\Delta}{m_q} \cos(qz) \delta_{q,0} \\
&\quad - \frac{N_c}{6(4\pi)^2} \left[q^4 \frac{\Delta^2(\Delta^2 + 2\rho^2)}{(\Delta^2 + \rho^2)^2} + 24q^2 \mu_I^2 \frac{\Delta^2 \rho^2}{(\Delta^2 + \rho^2)^2} \right] + V_{\text{fin}} + \dots \\
V_{\text{mat}} &= -N_c T \int_p \left\{ \log \left[1 + e^{-\beta(E_u^\pm - \mu)} \right] + \log \left[1 + e^{-\beta(E_d^\pm - \mu)} \right] \right. \\
&\quad \left. + \log \left[1 + e^{-\beta(E_{\bar{u}}^\pm + \mu)} \right] + \log \left[1 + e^{-\beta(E_{\bar{d}}^\pm + \mu)} \right] \right\} .
\end{aligned}$$

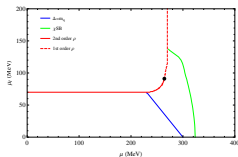
9. V_1 and V_{mat} must be independent of q in the limit $\Delta \rightarrow 0$.

1. Condensates at $T = 0$



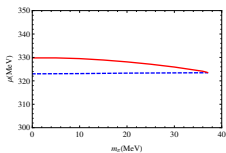
Chiral condensate (blue line) and pion condensate (red line) as functions of the isospin chemical potential μ_I for $\mu = 0$ and $\mu = 260$ MeV at the physical point and $T = 0$.

2. Phase diagram



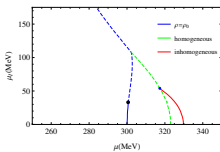
Phase diagram in the μ - μ_I plane at the physical point in the homogeneous case at $T = 0$.

3. Inhomogeneous condensate



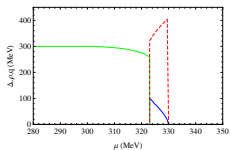
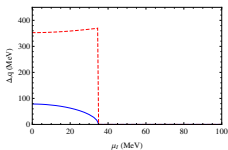
Lower and upper limits of the chemical potential μ where an inhomogeneous phase exists as a function of m_π for $\mu_I = 0$.

4. Inhomogeneous phases



Phase diagram in the μ - μ_I plane at $T = 0$ in the chiral limit. See main text for details.

5. Condensates



Chiral condensate Δ (blue line) and wave vector q (red line) as a function of the isospin chemical potential μ_I in the chiral limit for $\mu = 325$ MeV.

Chiral condensate (blue line), wave vector (red line), and pion condensate (green line) as functions of the chemical potential μ in the chiral limit for $\mu_I = 5$ MeV.