

Low energy nuclear systems: The two-nucleon system

Introduction

- It is more than 100 years that we have research in nuclear physics
- At present times it is a very active field of research
- Many of the motivations will be discussed at the school
- It is a strong interacting many-body system:
 - Very complicate to describe
 - Not only for discrete states but also continuum states
- The nuclear interaction is still under active research

Introduction

- It is more than 100 years that we have research in nuclear physics
- At present times it is a very active field of research
- Many of the motivations will be discussed at the school
- It is a strong interacting many-body system:
 - Very complicate to describe
 - Not only for discrete states but also continuum states
- The nuclear interaction is still under active research

Introduction

- It is more than 100 years that we have research in nuclear physics
- At present times it is a very active field of research
- Many of the motivations will be discussed at the school
- It is a strong interacting many-body system:
 - Very complicate to describe
 - Not only for discrete states but also continuum states
- The nuclear interaction is still under active research

The NN potential

The potential can depends on

- the distance $\longrightarrow V(r)$
- the spins angle $\longrightarrow V_\sigma(r) \vec{\sigma}_1 \cdot \vec{\sigma}_2$
- the spin and ℓ angle $\longrightarrow V_{ls}(r) \vec{\ell} \cdot \vec{S}$
- the spins and $\vec{r} = \vec{r}_1 - \vec{r}_2$ angles $\longrightarrow V_T(r) S_{12}$
with the tensor operator $S_{12} = 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$
- the different isospin channels $\longrightarrow \vec{\tau}_1 \cdot \vec{\tau}_2$
- $V(1, 2) = V(r) + V_\sigma(r) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_t(r) \vec{\tau}_1 \cdot \vec{\tau}_2 + V(r)_{\sigma\tau} (\vec{\sigma}_1 \cdot \vec{\sigma}_2) (\vec{\tau}_1 \cdot \vec{\tau}_2) + V_{ls}(r) \vec{\ell} \cdot \vec{S} + V_T(r) S_{12} + \dots$



The two-nucleon System

The two-nucleon wave function has total J , T and parity quantum numbers

$$\psi_{J^\pi}^T(1, 2) = \sum_{ls} \frac{u_{ls}(r)}{r} [Y_l(\hat{r}) \otimes \chi_s]_{JJ_z} \xi_{TT_z}$$

with

$$\chi_{ss_z} = \sum_{s_1, s_2} = \left(\frac{1}{2}s_1 \frac{1}{2}s_2 | ss_z\right) \chi_{\frac{1}{2}s_1}(1) \chi_{\frac{1}{2}s_2}(2)$$

$$\xi_{TT_z} = \sum_{t_1, t_2} = \left(\frac{1}{2}t_1 \frac{1}{2}t_2 | TT_z\right) \xi_{\frac{1}{2}t_1}(1) \xi_{\frac{1}{2}t_2}(2)$$

and

$$[Y_l(\hat{r}) \otimes \chi_s]_{JJ_z} = \sum_{l_z s_z} (l_z s_z | JJ_z) Y_{l_z}(\hat{r}) \chi_{ss_z}$$

The Two-Nucleon System

Antisymmetrization:

$$\psi_{J\pi}^T(1,2) = -\psi_{J\pi}^T(2,1) \Rightarrow l + s + T \equiv \text{odd}$$

→ *exercice*

The lower quantum numbers are:

l	s	T	$J\pi$	
0	0	1	0^+	
0,2	1	0	1^+	→ deuteron channel
1	0	0	1^-	
1	1	1	0^-	
1	1	1	1^-	
1,3	1	1	2^-	→ coupled channel
2	0	1	2^+	

Determination of the reduced wave function $u_{ls}(r)$

The structure of the two-nucleon system is governed by the NN force.
Let us look to the Schroedinger equation:

$$H\psi_{J\pi}^T = \left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2) \right] \psi_{J\pi}^T = E\psi_{J\pi}^T$$

Defining

$$\psi_{J\pi}^T(1,2) = \sum_{ls} \frac{u_{ls}(r)}{r} [Y_l(\hat{r}) \otimes \chi_s]_{JJ_z} \xi_{tt_z} = \sum_{ls} \frac{u_{ls}(r)}{r} |lsJ\rangle$$

At low energies only s-wave contributes in only two possible states:
 $J^\pi = 0^+$ and $J^\pi = 1^+$

$$\psi_{0^+}^1(1,2) = u_0(r)|000\rangle \text{ and } \psi_{1^+}^0(1,2) = u_1(r)|011\rangle$$

Zero-energy scattering

The Schroedinger equation is:

$$\left[-\frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + V_s(r) \right] u_s(r) = E u_s(r)$$

If the potential is the same, the low-energy behavior of the $s = 0, 1$ states would be equal. However a bound state appears for $s = 1$ and not for $s = 0 \rightarrow$ the potential is different in the two spin channels.

Furthermore, let us look at the zero-energy state:

$$\left[-\frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + V_s(r) \right] u_s(r) = 0$$

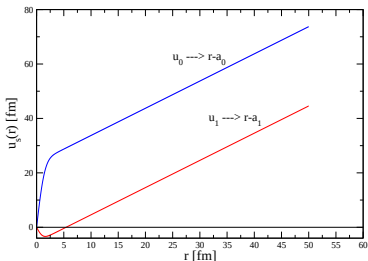
The nuclear potential is short-range, $V(r > r_N) = 0$, and for $r > r_N$, the equation is

$$\frac{\partial^2}{\partial r^2} u_s(r) = 0$$

Therefore

$$u_s(r) \rightarrow r - a_s$$

with a_s the scattering length in spin channel s . The equation $u_s''(r) - V_s(r)/(\hbar^2/m) = 0$ is integrated from the origin $u_s(0) = 0$ up to matching the above linear behavior.



Experimental data for these quantities are: $a_0 = -23.74(2)$ fm and $a_1 = 5.42(1)$ fm, a further evidence of the different strength of the potential in channels $s = 0, 1$.

Positive energies

At positive energies the Schroedinger equation is:

$$\left[-\frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + V_s(r) \right] u_s(r) = E u_s(r)$$

or

$$u_s''(r) + \left[k^2 - \frac{mV_s(r)}{\hbar^2} \right] u_s(r) = 0$$

with

$$u_s(r > r_N) = \sin(kr) + R \cos(kr)$$

In general

$$u_s(r > r_N) = \begin{cases} \sin(kr) + R \cos(kr) & K - \text{matrix} \\ e^{-ikr} - S e^{ikr} & S - \text{matrix} \\ \sin kr + T e^{ikr} & T - \text{matrix} \\ R^{-1} \sin(kr) + \cos(kr) & K^{-1} - \text{matrix} \end{cases}$$

The Phase-shift δ

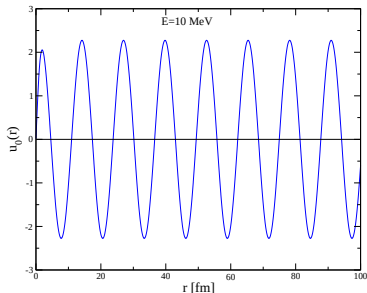
For single channels: $R = \tan \delta$ with δ the phase-shift. And

$$S = e^{2i\delta}; T = \sin \delta e^{i\delta}$$

To determine the phase-shift we impose the boundary conditions:

1) $u_s(0) = 0$

2) $u_s(r > r_N) = \sin(kr) + R \cos(kr)$



Effective Range expansion

The scattering length and the phase-shift are related through the effective range expansion, valid at low energies:

$$k \cot \delta = -\frac{1}{a_s} + \frac{1}{2}R_s k^2 \quad \rightarrow \text{exercise}$$

where $\hbar^2 k^2 / m = E$ is the energy of the process and R_s is the effective range in channel s :

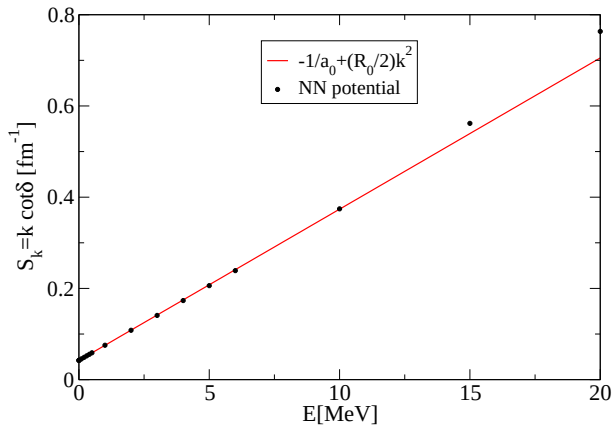
$$R_s = 2 \int_0^\infty \left[\phi_s^2 - u_s^2(k=0) \right] dr$$

with $\phi_s = 1 - r/a_s$ and $u_s(k=0)$ the zero-energy solution normalized in such a way that

$$u_s(r > r_N) \rightarrow 1 - r/a_s$$

In the above integral, the integrand goes to zero very quickly as ϕ_s and $u_s(k=0)$ becomes equal.

Effective Range expansion



np effective range

For s-wave scattering the equation at two different energies

$$\frac{\partial^2 u_1(r)}{\partial r^2} - \left(\frac{mV(r)}{\hbar^2} - k_1^2 \right) u_1(r) = 0$$

$$\frac{\partial^2 u_2(r)}{\partial r^2} - \left(\frac{mV(r)}{\hbar^2} - k_2^2 \right) u_2(r) = 0$$

subtracting and integrating the equations the equations we have

$$(u_2 u_1' - u_1 u_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} u_1 u_2 dr$$

or using the asymptotic form $\phi_i = [\cos(k_i r) + \cot \delta \sin(k_i r)]$

$$(\phi_2 \phi_1' - \phi_1 \phi_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} \phi_1 \phi_2 dr$$

np effective range

making $r' > r_N$ and subtracting the two equations, the asymptotic part is canceled. For $r \rightarrow 0$ the wave functions $u_1, u_2 \rightarrow 0$. The behavior of the functions ϕ_i as $r \rightarrow 0$ are $\phi_1(0) = \phi_2(0) = 1$ and

$$\phi_i'(0) = k_i \cot \delta_i$$

The only remaining term is

$$\phi_2 \phi_1' - \phi_1 \phi_2' \xrightarrow{r \rightarrow 0} k_2 \cot \delta_2 - k_1 \cot \delta_1 = (k_2^2 - k_1^2) \int_0^\infty (\phi_1 \phi_2 - u_1 u_2)$$

for $k_1 \rightarrow 0$ and generic k

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} R_s k^2$$

The pole of the S-matrix

The S-matrix is defined as

$$S = e^{2i\delta} = \frac{e^{i\delta}}{e^{-i\delta}} = \frac{\cos\delta + i\sin\delta}{\cos\delta - i\sin\delta} = \frac{\cot\delta + i}{\cot\delta - i}$$

and $k\cot\delta = ik$ is a pole of the S-matrix. The extension to the complex plane (at the imaginary axis) corresponds to bound states: $k = i\kappa_d$. Resulting in $i\kappa_d\cot\delta = -\kappa_d$. Using the effective range expansion, valid only for bound states close to the threshold (shallow states), we have:

$$\kappa_d = \frac{1}{a_s} + \frac{1}{2}R_s\kappa_d^2$$

from where it would be possible to extract the bound state energy using scattering properties.

Is the deuteron a shallow bound state?

Experimental data:

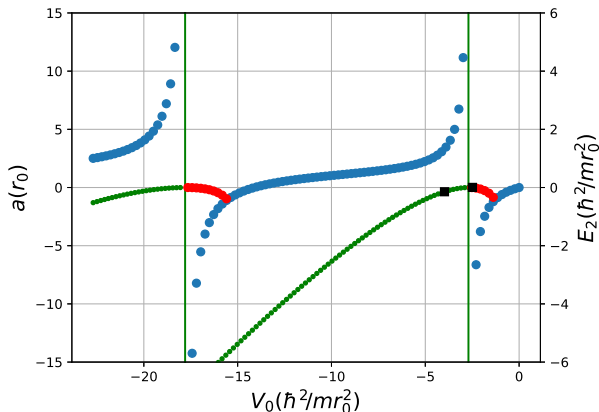
np	a_1	5.419(7) fm
	R_1	1.753(8) fm
	a_0	-23.740(20) fm
	R_0	2.77(5) fm
	E_d	2.22456 MeV
nn	a_0	-18.90(40) fm
	R_0	2.75(11) fm
pp	a_0	-7.8063(26) fm
	R_0	2.794(14) fm

Using these data the deuteron binding energy can be estimated as:

$$\kappa_d = \frac{1}{R_1} \left(1 - \sqrt{1 - 2R_1/a_1} \right)$$

$$E_d = \hbar^2 \kappa_d^2 / m = 2.223 \text{ MeV.}$$

The sign of the scattering length



The black squares represent the deuteron and nn virtual state

pp scattering

Two protons have isospin $T = 1$, the wave function is

$$\psi_{J\pi}^1(1, 2) = \sum_{ls} \frac{u_{ls}(r)}{r} [Y_l(\hat{r}) \otimes \chi_s]_{JJ_z} \xi_{11}$$

Outside the nuclear range the equations are not coupled. For energy $E = \hbar^2 k^2 / m$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} - \frac{me^2}{\hbar^2 r} + k^2 \right) u_{ls}(r) = 0$$

defining the dimensionless variable $z = kr$

$$u_{ls}''(z) + \left(1 - \frac{2\eta}{z} - \frac{l(l+1)}{z^2} \right) u_{ls}(z) = 0$$

with $2\eta = (\hbar^2/m)e^2/k$ or $2k\eta = (\hbar^2/m)e^2 = 1/R_p$ with R_p the protonic Bohr radius

pp scattering

The solutions are combinations of the regular and irregular Coulomb wave functions $F_l(\eta, z)$, $G_l(\eta, z)$ with the following asymptotic behavior

$$F_l(\eta, z) \xrightarrow{z \rightarrow \infty} \sin(z - \eta \ln 2z - l\pi/2 + \sigma_l)$$

$$G_l(\eta, z) \xrightarrow{z \rightarrow \infty} \cos(z - \eta \ln 2z - l\pi/2 + \sigma_l)$$

with the Coulomb phase-shift $\sigma_l = \arg \Gamma(l + 1 + i\eta)$

The asymptotic form of the wave function is the combination

$$u_{ls}(r > r_N) \propto F_l(\eta, z) + \tan \delta G_l(\eta, z)$$

with δ the nuclear phase-shift with respect to the Coulomb field

pp effective range

The effective range formula can be obtained similar to the *np* case. For s-wave scattering the equation at two different energies

$$\frac{\partial^2 u_1(r)}{\partial r^2} - \left(\frac{me^2}{\hbar^2 r} + \frac{mV(r)}{\hbar^2} - k_1^2 \right) u_1(r) = 0$$

$$\frac{\partial^2 u_2(r)}{\partial r^2} - \left(\frac{me^2}{\hbar^2 r} + \frac{mV(r)}{\hbar^2} - k_2^2 \right) u_2(r) = 0$$

subtracting and integrating the equations the equations we have

$$(u_2 u_1' - u_1 u_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} u_1 u_2 dr$$

or using the asymptotic form $\phi_i = C_{0i}[G_0(k_i r) + \cot \delta F_0(k_i r)]$

$$(\phi_2 \phi_1' - \phi_1 \phi_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} \phi_1 \phi_2 dr$$

pp effective range

making $r' > r_N$ and subtracting the two equations, the asymptotic part is canceled. For $r \rightarrow 0$ the wave functions $u_1, u_2 \rightarrow 0$. The behavior of the functions ϕ_i as $r \rightarrow 0$ can be obtained from

$$F_0(z) \xrightarrow{z \rightarrow 0} C_0 z (1 + \eta z + \dots)$$

$$G_0(z) \xrightarrow{z \rightarrow 0} \frac{1}{C_0} \{1 + 2\eta z [\ln(2\eta z) + 2\gamma - 1 + h(\eta)] + \dots\}$$

with $\gamma = 0.577215\dots$ the Euler constant, $C_0 = 2\pi\eta/(e^{2\pi\eta} - 1)$ and $h(\eta) = -\ln \eta + \operatorname{Re}\psi(1 + i\eta)$, with ψ denoting the digamma function. The only remaining term is

$$\phi_2 \phi_1' - \phi_1 \phi_2' \xrightarrow{r \rightarrow 0} \frac{1}{R_p} [h(\eta_2) - h(\eta_1)] + C_0^2 k_2 \cot \delta_2 - C_0^2 k_1 \cot \delta_1$$

pp effective range

For $k_1 \rightarrow 0$ we define

$$\lim_{k \rightarrow 0} C_0^2 k \cot \delta(k) + \frac{1}{R_p} h(\eta) = \frac{-1}{a_{pp}}$$

and the effective range expansion results

$$C_0^2 k \cot \delta(k) + \frac{1}{R_p} h(\eta) = \frac{-1}{a_{pp}} + k^2 \int_0^\infty (\phi \phi_0 - u u_0) dr$$

defining again

$$R_0 = 2 \int_0^\infty (\phi_0^2 - u_0^2) dr$$

the effective range expansion is

$$C_0^2 k \cot \delta(k) + \frac{1}{R_p} h(\eta) \approx \frac{-1}{a_{pp}} + \frac{1}{2} k^2 R_0$$

Relation between pp and nn effective ranges

Taking the proper limits, the nn scattering length in the same nuclear potential is related to the pp scattering length by the following approximate relation

$$\frac{1}{a_{pp}} = \frac{1}{a_{nn}} + \frac{1}{R_p} \left(\ln \frac{R_0}{R_p} + 0.330 \right)$$

where the number 0.330 is obtained from a combination of the Euler constant, $\ln \pi$ and the cosine integral $C_i(\pi)$.

Remembering the experimental values: $a_{pp} = -7.8063(26)$ fm and $R_0 = 2.794(14)$ fm. We can calculate the corrected value: $\tilde{a}_{pp} = -17.137$ fm. To be compared to the $a_{nn} = -18.90(40)$ fm.

There is a small difference between the pp and the nn forces.