

Low energy nuclear physics: calculating observables

The two-nucleon system with a realistic potential

a) The deuteron ($J^\pi = 1^+$, $s = 1$, $t = 0$)

$$\psi_{1^+}^0(1, 2) = \sum_l \frac{u_l(r)}{r} [Y_l(\hat{r}) \otimes \chi_1]_{1M} \xi_{00} = \frac{u_0}{r} |011\rangle + \frac{u_2}{r} |211\rangle$$

The two components u_0 and u_2 are determined by the potential in $s, t = 1, 0$ channel:

$$V_{10}(1, 2) = V_c(r) + V_{ls}(r) \vec{\ell} \cdot \vec{S} + V_T(r) S_{12} + V_{l2}(r) \ell^2 + V_{ls2}(r) (\vec{\ell} \cdot \vec{S})^2 + \dots$$

Considering these potential terms, the equation for the radial components is

$$(H - E) \left[\frac{u_0}{r} |011\rangle + \frac{u_2}{r} |211\rangle \right] = 0$$

deuteron equations

Projecting the equations in the two angular states

$$\begin{cases} \left[-\frac{\hbar^2}{m} \frac{\partial^2}{\partial r^2} + \langle 011 | V(1,2) | 011 \rangle - E \right] \frac{u_0}{r} = -\langle 011 | V(1,2) | 211 \rangle \frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m} \left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2} \right) + \langle 211 | V(1,2) | 211 \rangle - E \right] \frac{u_2}{r} = -\langle 211 | V(1,2) | 011 \rangle \frac{u_0}{r} \end{cases}$$

The next step is to calculate the matrix elements of the potential

$$\langle 011 | V(1,2) | 011 \rangle = V_c(r)$$

$$\langle 211 | V(1,2) | 011 \rangle = \sqrt{8} V_T(r)$$

$$\langle 211 | V(1,2) | 211 \rangle = V_c(r) - 3V_{ls}(r) + 9V_{ls2} + 6V_{l2}(r) - 2V_T(r)$$

Finally we apply the boundary conditions ($r \rightarrow \infty$) :

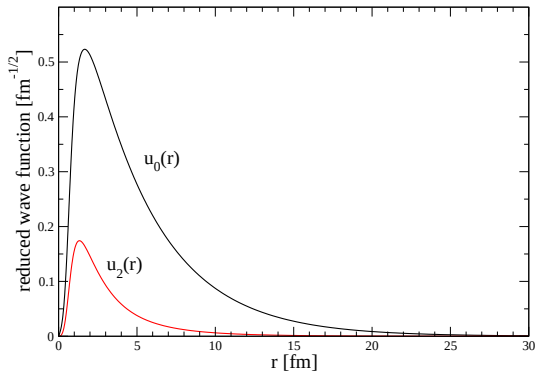
$$(\partial^2 / \partial r^2 - k_d^2) u_0(r) = 0$$

$$u_0(r) \rightarrow e^{-k_d r}$$

$$(\partial^2 / \partial r^2 - 6/r^2 - k_d^2) u_2(r) = 0$$

$$u_2(r) \rightarrow e^{-k_d r} (1 + 3/k_d r + 3/k_d^2 r^2)$$

Deuteron wave function



with the normalization $\int_0^\infty (u_0^2 + u_2^2) dr = 1$

and a D -state probability $P_d = 100 \times \int_0^\infty u_2^2 dr \approx 5\%$

Evidence for the D -state of the deuteron

Evidences for the deuteron D -state are coming from the quadrupole moment Q (different from zero) and from the magnetic moment (different from summing the proton and deuteron magnetic moments).

A) The quadrupole moment:

$$Q = \left(\frac{16\pi}{5} \right)^{1/2} \int \Psi_{JJ}^* Q_{20} \Psi_{JJ} d^3\vec{r}$$

where

$$Q_{20} = e \sum_{k=1}^A r_k^2 Y_{20}(\theta_k) \left(\frac{1}{2} + t_3^k \right)$$

and the deuteron wave function written as

$$\Psi_{JJ} = \cos \epsilon u_0(r) |011\rangle + \sin \epsilon u_2(r) |211\rangle$$

Evidence for the D -state of the deuteron

The $\ell = 0$ component does not contribute

$$Q = \left(\frac{16\pi}{5}\right)^{1/2} \frac{e}{\sqrt{4\pi}} \cos \epsilon \sin \epsilon (2011|11) \int_0^\infty u_0(r)u_2(r)r^2 dr + \sin^2 \epsilon \dots$$

the experimental value is $Q = 0.2860(15) \text{ fm}^2$. This results can be reproduced with $\sin^2 \epsilon \approx 0.04$ (a D -state probability of about 4%).

B) The magnetic moment μ :

Is the mean value of the z -component of the magnetic moment operator

$$\vec{M} = \mu_0 \left(\sum_k^A \left(\frac{1}{2} + t_3^k \right) (\vec{\ell}_k + g_k^p \vec{s}_k) + \sum_k^A \left(\frac{1}{2} - t_3^k \right) g_k^n \vec{s}_k \right)$$

and the magnetic moment is

$$\mu = \langle \Psi_{JJ} | M_z | \Psi_{JJ} \rangle$$

Evidence for the D -state of the deuteron

the gyroscope factors $g^p/2 = 2.792782(17)$, $g^n/2 = -1.913148(66)$. Defining the scalar $\mu_s = (g^p + g^n)/2$ and vector $\mu_v = (g^p - g^n)/2$ moments, the operator is

$$\vec{M} = \mu_0 \left(\sum_k^A \left(\frac{1}{2} + t_3^k \right) \vec{\ell}_k + \mu_s \vec{S} + \mu_v \sum_k^A t_3^k \vec{S}_k \right)$$

which for the deuteron reduces to $(\mu_0/2)\vec{\ell} + \mu_s \vec{S}$. Making some algebra (**exercise**) and considering the deuteron angular momentum $J = 1$, the magnetic moment results

$$\mu_d/\mu_0 = \mu_s + \sin^2 \epsilon \left(\frac{3}{4} - \frac{\mu_s}{2} \right) = 0.879 - 0.569 \sin^2 \epsilon$$

the measured value is $\mu_d/\mu_0 = 0.857406(1)$, implying a D -state probability of around 4%.

Coupled equations in the continuum

We consider the following potential in each in s, t channel:

$$V_{st}(1, 2) = V_c(r) + V_{ls}(r)\vec{\ell} \cdot \vec{s} + V_T(r)S_{12} + V_{l2}(r)\ell^2 + V_{ls2}(r)(\vec{\ell} \cdot \vec{s})^2$$

For spin single channels the potential reduced to

$$V_{0t}(1, 2) = V_c(r) + V_{l2}(r)\ell^2$$

and the equation are not coupled

$$\psi_{Jt}^{\pi}(1, 2) = \frac{u_J(r)}{r} [Y_l(\hat{r}) \otimes \chi_0]_{JM} \xi_{tt_z} = \frac{u_J}{r} Y_{JM} \chi_{00} \xi_{tt_z} = \frac{u_J}{r} |J0J\rangle$$

The equation to be solved is

$$\left(-\frac{\hbar^2}{m} \nabla^2 + V(1, 2) - E \right) \frac{u_J(r)}{r} |J0J\rangle$$

considering $\langle J0J | V(1, 2) | J0J \rangle = V_c(r) + J(J+1)V_{l2}(r)$ we have

$$\left[-\frac{\hbar^2}{m} \left(\frac{\partial^2}{\partial r^2} - \frac{J(J+1)}{r^2} \right) + V_c(r) + J(J+1)V_{l2}(r) - E \right] u_J(r) = 0$$

spin single channel equation

$$u_J''(r) - \left[\frac{J(J+1)}{r^2} + \frac{V_c(r) + J(J+1)V_{l2}(r)}{\hbar^2/m} - k^2 \right] u_J(r) = 0$$

For spin single channels there are not bound states. We consider continuum states. Asymptotically we have

$$u_J''(r) - \left[\frac{J(J+1)}{r^2} - k^2 \right] u_J(r) = 0$$

and

$$u_J(r \rightarrow \infty) \longrightarrow r[j_J(kr) + \tan \delta_J \mathcal{Y}_J(kr)]$$

where $j_J(kr)$, $\mathcal{Y}_J$ are Bessel functions of fractional order. At the origin ($r \rightarrow 0$) the reduced function behaves as $u_J \rightarrow r^{J+1}$. In the case of two protons ($t = 1$ and ℓ even) the asymptotic part is

$$u_J''(r) - \left[\frac{J(J+1)}{r^2} - \frac{me^2}{\hbar^2 r} - k^2 \right] u_J(r) = 0$$

with $u_J(r \rightarrow \infty) \longrightarrow F_J(\eta, \rho) + \tan \delta_J G_J(\eta, \rho)$

Matrix elements of the potential $V(1, 2)$

In general we have to compute

$$\langle LSJ|V(1, 2)|L'S'J \rangle = \delta_{SS'} \langle LSJ|V(1, 2)|L'SJ \rangle$$

The tensor operator S_{12} acts in $S = 1$ and can couple different L states:

$$\langle J \ 1J|S_{12}|J \ 1J \rangle = 2$$

$$\langle J - 1 \ 1J - 1|S_{12}|J - 1 \ 1J - 1 \rangle = \frac{-2(J-1)}{2J+1}$$

$$\langle J + 1 \ 1J|S_{12}|J - 1 \ 1J \rangle = \frac{6\sqrt{J(J+1)}}{2J+1}$$

$$\langle J + 1 \ 1J|S_{12}|J + 1 \ 1J \rangle = \frac{-2(J+2)}{2J+1}$$

exercise!

Remember: $S_{12} = 3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2$

Matrix elements of the potential $V(1, 2)$

The central potential is very simple

$$\langle LSJ|V(1, 2)|L'S'J \rangle = \delta_{SS'}\delta_{LL'}V_c(r)$$

The spin-orbit interaction is

$$\langle LSJ|\vec{L} \cdot \vec{S}|L'S'J \rangle = \delta_{SS'}\delta_{LL'}\frac{1}{2}[J(J+1) - L(L+1) - S(S+1)]$$

and the L^2 interactions is

$$\langle LSJ|L^2|L'S'J \rangle = \delta_{SS'}\delta_{LL'}L(L+1)$$

spin $s = 1$ equations

In the case in which $l = J$ there are not coupled equations (why?). The equation is similar to the $s = 0$ case but the matrix elements of the potential are:

$$\langle J1J|V(1,2)|J1J \rangle = V_c(r) - V_{ls}(r) + V_{ls2} + J(J+1)V_{l2}(r) + 2V_T(r)$$

For $\ell = J \pm 1$ they are coupled equations

$$\begin{cases} \left(\frac{\partial^2}{\partial r^2} - \frac{(J-1)J}{r^2} - \frac{V_{J-1,J-1}}{\hbar^2/m} + k^2 \right) u_{J-1} = \frac{V_{J-1,J+1}}{\hbar^2/m} u_{J+1}(r) \\ \left(\frac{\partial^2}{\partial r^2} - \frac{(J+1)(J+2)}{r^2} - \frac{V_{J+1,J+1}}{\hbar^2/m} + k^2 \right) u_{J+1} = \frac{V_{J+1,J-1}}{\hbar^2/m} u_{J-1}(r) \end{cases}$$

where we have defined

$$V_{J-1,J-1} = \langle J-11J|V(1,2)|J-11J \rangle$$

$$V_{J-1,J+1} = V_{J+1,J-1} = 6 \frac{\sqrt{J(J+1)}}{2(J+1)}$$

$$V_{J+1,J+1} = \langle J+11J|V(1,2)|J+11J \rangle$$

Asymptotic region

The coupled equations are not coupled asymptotically

$$\begin{cases} \left(\frac{\partial^2}{\partial r^2} - \frac{(J-1)J}{r^2} + k^2 \right) u_{J-1} = 0 \\ \left(\frac{\partial^2}{\partial r^2} - \frac{(J+1)(J+2)}{r^2} + k^2 \right) u_{J+1} = 0 \end{cases}$$

It is possible to construct two different asymptotic configurations

a) $u_{J-1}^a(r \rightarrow \infty) = r [j_{J-1}(kr) + \mathcal{R}_{J-1,J-1}\mathcal{Y}_{J-1}]$

$$u_{J+1}^a(r \rightarrow \infty) = r [\mathcal{R}_{J-1,J+1}\mathcal{Y}_{J+1}]$$

b) $u_{J-1}^b(r \rightarrow \infty) = r [\mathcal{R}_{J+1,J-1}\mathcal{Y}_{J-1}]$

$$u_{J+1}^b(r \rightarrow \infty) = r [j_{J+1}(kr) + \mathcal{R}_{J+1,J+1}\mathcal{Y}_{J+1}]$$

Introducing the S-matrix

The relative weights with respect to the regular solutions form the reactance matrix (or $\mathcal{R}$ -matrix). The solutions can be combined to form the ingoing or outgoing waves.

$$a) \quad u_{J-1}^0(r \rightarrow \infty) = r [\mathcal{I}_{J-1}(kr) - S_{J-1,J-1} \mathcal{O}_{J-1}]$$

$$u_{J+1}^0(r \rightarrow \infty) = r [S_{J-1,J+1} \mathcal{O}_{J+1}]$$

$$b) \quad u_{J-1}^1(r \rightarrow \infty) = r [S_{J+1,J-1} \mathcal{O}_{J-1}]$$

$$u_{J+1}^1(r \rightarrow \infty) = r [\mathcal{I}_{J+1}(kr) - S_{J+1,J+1} \mathcal{O}_{J+1}]$$

With an incident wave in $|J-1, 1, J\rangle$ state, the S-matrix elements give the probability of having an outgoing wave in states $|J-1, 1, J\rangle$ and $|J+1, 1, J\rangle$. To be noticed that the tensor force is the only component of the force that induces a change in the orbital angular momentum

Phase-shift and mixing parameters

For single channels the S -matrix can be defined as $S = e^{2i\delta}$ For coupled channels the S -matrix is a 2×2 matrix. In general the following properties are verified: $SS^\dagger = 1$ and $S_{ij} = S_{ji}$. It can be put in the following way:

$$S = U^{-1}e^{2i\Delta}U$$

with

$$U = \begin{pmatrix} \cos \epsilon_J & \sin \epsilon_J \\ -\sin \epsilon_J & \cos \epsilon_J \end{pmatrix} ; e^{2i\Delta} = \begin{pmatrix} e^{2i\delta_{J\alpha}} & 0 \\ 0 & e^{2i\delta_{J\beta}} \end{pmatrix}$$

This is called the Blatt and Biedenharn decomposition, mostly used.

Constructing the NN differential cross section

The incident wave function is determined by the incident nucleon momentum, spin, isospin projections, and by the target quantum numbers

$$e^{ikz} \chi_{1/2\mu_1} \xi_{1/2t_1} \chi_{1/2\mu_2} \xi_{1/2t_2}$$

It is convenient to construct states with total spin s and total isospin t

$$\chi_{s\mu} = \sum_{\mu_1\mu_2} (1/2\ \mu_1\ 1/2\ \mu_2 | s\ \mu) \chi_{1/2\mu_1}(1) \chi_{1/2\mu_2}(2)$$

$$\xi_{TT_z} = \sum_{t_1t_2} (1/2\ t_1\ 1/2\ t_2 | T\ T_z) \xi_{1/2t_1}(1) \xi_{1/2t_2}(2)$$

With the incident wave

$$\sum_{s\mu} \sum_{tt_z} (1/2\ \mu_1\ 1/2\ \mu_2 | s\ \mu) (1/2\ t_1\ 1/2\ t_2 | t\ t_z) e^{ikz} \chi_{s\mu} \xi_{TT_z}$$

Constructing the NN differential cross section

Let us concentrate for the moment in the case $s = 1$ and expands the free wave in partial waves

$$e^{ikz} \chi_{1\mu} \xi_{TT_z} = \sum_{\ell=0}^{\infty} \sum_{J=\ell-1}^{\ell+1} \sqrt{4\pi(2\ell+1)} i^{\ell} j_{\ell}(kr) (\ell 0 1 \mu | J \mu) [Y_{\ell}(\theta, 0) \otimes \chi_1]_{J\mu} \xi_{TT_z}$$

Now we have to remember the asymptotic form of the wave function after solving the Schroedinger equation

$$\psi_{J\pi}^T(r \rightarrow \infty) \longrightarrow \left\{ j_{\ell}(kr) [Y_{\ell} \otimes \chi_1]_{JJ_z} + \sum_{\ell'} T_{\ell\ell'}^J \mathcal{O}_{\ell'}(kr) [Y_{\ell'} \otimes \chi_1]_{JJ_z} \right\} \xi_{TT_z}$$

where I have used the combination of the regular solution relative to the outgoing given by the T -matrix.

The wave function is multiplied by the proper factors and combined to form the regular term of the incident wave function.

The scattering matrix

$$\rightarrow e^{ikz} \chi_{1\mu} \xi_{TT_z} + \sum_{J=0}^{\infty} \sum_{\ell\ell'} \sqrt{4\pi(2\ell+1)} i^{\ell} (\ell 0 1 \mu | J \mu) T_{\ell\ell'}^J \mathcal{O}_{\ell'}(kr) [Y_{\ell'} \otimes \chi_1]_{J\mu} \xi_{TT_z}$$

The outgoing spherical wave behaves as $\mathcal{O}_{\ell} \rightarrow (1/kr) e^{i(kr - \pi\ell/2)}$

$$\psi \rightarrow e^{ikz} \chi_{1\mu} \xi_{TT_z} + \sum_{\mu'} M_{\mu\mu'}(\theta) \frac{e^{ikr}}{r} \chi_{1\mu'} \xi_{TT_z}$$

where we have introduced the scattering matrix

$$M_{\mu\mu'}(\theta) = \frac{\sqrt{4\pi}}{k} \sum_{J=0}^{\infty} \sum_{\ell\ell'} \sqrt{2\ell+1} (\ell 0 1 \mu | J \mu) (\ell m' 1 \mu' | J \mu) T_{\ell\ell'}^J Y_{\ell'm'}(\theta, 0)$$

with $m' = \mu - \mu'$.

The scattering matrix

The result can be generalized to $s = 0, 1$ as

$$M_{\mu\mu'}^{ss'} = \frac{\sqrt{4\pi}}{k} \sum_{J=0}^{\infty} \sum_{\ell\ell'} \sqrt{2\ell+1} (\ell 0 s \mu | J \mu) (\ell m' s' \mu' | J \mu) T_{\ell\ell'}^{ss'} Y_{\ell'm'}(\theta, 0)$$

In the case of NN scattering

$$T_{\ell\ell'}^{ss'} = \delta_{ss'} T_{\ell\ell'}^{s,J}$$

is 2×2 (for $s = 1$ and $\ell = J \pm 1$) or a 1×1 (for $s = 0$ or $s = 1$ and $\ell = J$). In conclusion the scattering matrix $M_{\mu\mu'}^{ss'}$ is a 4×4 matrix

$$M_{\mu\mu'}^{ss'} = \delta_{ss'} M_{\mu\mu'}^s$$

from which the cross section and scattering observables can be obtained

$$\sigma(\theta) = \frac{1}{4} \text{tr}(MM^\dagger) ; \quad P(\theta) = \text{tr}(MM^\dagger P)$$

The pp scattering matrix

In the case of two protons the scattering matrix results

$$M_{\mu\mu'}^{ss'} = \delta_{ss'} \frac{\sqrt{4\pi}}{k} \left\{ f_c(\theta) \delta_{\mu\mu'} + \sum_{J\ell\ell'} \sqrt{2\ell+1} (\ell 0 s \mu | J \mu) (\ell m' s' \mu' | J \mu) e^{i\sigma_\ell} T_{\ell\ell'}^{ss'} e^{i\sigma_{\ell'}} Y_{\ell'm'}(\theta, 0) \right\}$$

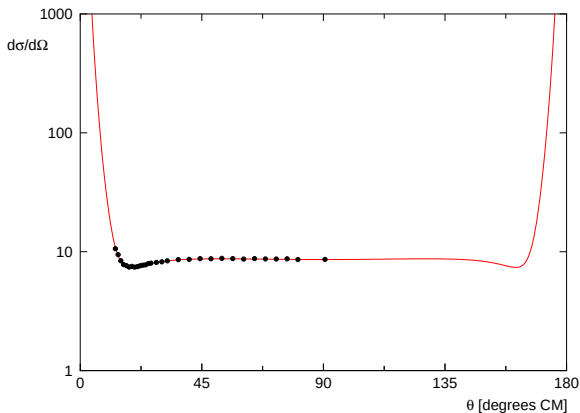
with $f_c(\theta)$ the Coulomb amplitude and σ_ℓ the Coulomb phase-shift. Recalling that

$$F_\ell(\rho \rightarrow \infty) \longrightarrow \sin(\rho - \eta \ln 2\rho - \ell\pi/2 + \sigma_\ell)$$

and $\sigma_\ell = \arg\Gamma(\ell + 1 + i\eta)$

$$f_c(\theta) = \frac{1}{4k^2 R_p} \operatorname{cosec}(\theta/2) \exp(-2i\eta \ln \sin \theta/2 + i\pi + 2i\sigma_0)$$

pp differential cross sections

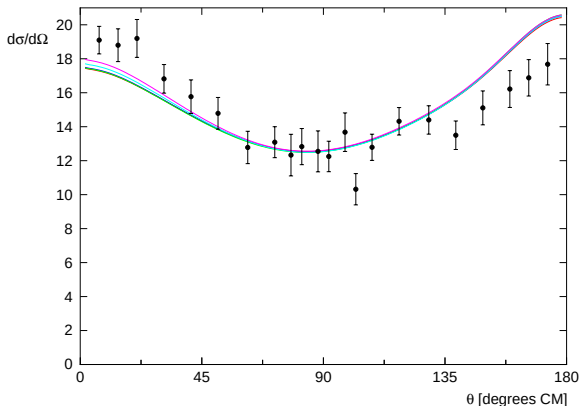


pp observable $d\sigma/d\Omega$ at $T_{\text{lab}} = 49.41$ MeV

— NijmI potential

• BA67

np differential cross sections



np observable $d\sigma/d\Omega$ at $T_{\text{lab}} = 47.5$ MeV

- PWA93
- NijmI potential
- NijmII potential
- Reid93 potential
- Nijm93 potential
- SC63